Development and validation of a train-bridge interaction model

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Sharon Deceuninck
Gent, June 8, 2023
This master’s dissertation is part of an exam. Any comments formulated by the assessment committee during the oral presentation of the master’s dissertation are not included in this text.
Preface

The completion of this master dissertation would not have been possible without the invaluable support of numerous individuals who have played a crucial role throughout the year.

First and foremost, I express my gratitude to my supervisors, Prof. Dr. Ir. Wim De Waele and Prof. Dr. Ir. Mia Loccufier. Their expert guidance and advice have been instrumental in shaping the outcome of this research. I am grateful for their support, providing me with insightful feedback and constructive criticism, which has enabled me to deliver a scientifically relevant dissertation.

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In addition, I would like to express my sincere gratitude to my family and boyfriend for their support during my entire academic journey. Their enduring belief in me, coupled with their patient ears and interest in my master dissertation, have been a constant source of motivation. Even during the challenging moments, they stood by my side, providing the encouragement I needed to overcome obstacles and persevere.

Lastly, I would like to thank my wonderful friends for their support and for making my five years in Gent truly unforgettable.

To all those mentioned above, and to everyone who has contributed in any way to this endeavor, I extend my heartfelt thanks. Your assistance and encouragement have been indispensable, and I am truly grateful for your commitment.

Sharon Deceuninck
Gent, 8 juni 2023
Abstract

This dissertation presents the development of a dynamic train-bridge interaction model to determine the behavior of a Railway bridge during the passage of a train. First, the importance of including dynamic loads is mentioned. Secondly, an overview of the evolution in train modelling is given and the goal is to find a balance between model complexity and accuracy gain with respect to the desired results. Further, different solution methods are discussed to solve the dynamic train-bridge interaction.

In the following step, the direct integration method and the intersystem iteration method are developed and validated for different cases. Both models are simulated with the finite element software Abaqus. Because of the complexity, only the vertical wheel-rail interaction is considered. The first case looks at a simply supported beam under one moving spring-mass. Due to the simplicity, an analytical solution is found to validate the first model. The next cases simulate the passage of whole trains as a series of spring-damper-mass models and are compared with the simulation results found in the literature.

In the end, the usefulness of both methods for engineers is discussed. The methods are compared to a more narrow application wherefore it serves. This results in two useful methods, where the direct integration method is preferred for a general first attempt at the behavior of a railway bridge under the passage of a train. The intersystem iteration method is more computationally intensive, but gives more accurate results in the prediction of the structural health of the railway bridge.

Key words: Wheel-rail interaction, direct integration method, intersystem iteration method, vertical deformation
Development and validation of a train-bridge interaction model

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Abstract—In this paper, a wheel-rail interaction model is presented to simulate the behavior of a bridge while passing a train. Two different ways of modelling are discussed: the direct integration method (DIM) and the intersystem iteration method (ISIM). The simulations are performed with the finite element software Abaqus. First, the simplest case of a simply supported bridge under a moving spring-mass model is validated with the analytical solution. Subsequently, incorporating damping into the model adds an extra layer of complexity, rendering the search for analytical solutions unfeasible. The validation of the more complex models is accomplished by comparing them to similar simulations documented in the existing literature. The main focus lies on finding an efficient model that could be used by engineers to assess the structural condition of railway bridges. The dynamic interaction model gives a more accurate prediction of the behavior of the bridge under the passage of the train than including the dynamic amplification factor in the static calculation.

Index Terms—wheel-rail interaction, direct integration method, intersystem iteration method

I. INTRODUCTION

In the aim of the European project 'Sustainable bridges' [1], sustainability becomes more important in the (re)construction of railway bridges. The goal is to have a modal shift in the use of transport and stimulate people to prefer public transport or the bicycle instead of the car. Therefore, public transport options must become more attractive. Especially for rail transport, the capacity and the number of trains should be increased. The focus of the European project lies on the Railway bridges where it is not directly possible to enlarge the railway network, so the quantity of train passage will be enlarged. By increasing the capacity of the trains, the axle loads will become larger and could result in a shorter end-of-life span of the railway bridges. To guarantee the safety of the design, measurements and predictions of the railway bridges must be performed to have knowledge about the behavior of the railway bridges. Therefore, it becomes interesting to be able to predict the behavior of a bridge under the passage of a train.

II. IMPORTANCE OF A DYNAMIC INTERACTION MODEL

The research of dynamic wheel-rail interaction models dates back to 1949 when Willis and Stokes [2] perform real experiments of a moving mass over a simply supported beam. In their study, they compared the vertical deflection at the midspan of the beam, for various velocities $V$ of the mass, with the vertical deflection caused by a beam loaded with a static point load at the midspan. Figure 1 illustrates the vertical deflection at the midspan observed in the experiment. The static equilibrium curve gives the deformation at midspan due to a static point load. The other curves give the vertical deflection at midspan for different values of $\beta$, where $2a$ is the length of the beam and $S$ is the static deflection at midspan.

$$\beta = \frac{ga^2}{4V^2S} \quad (1)$$

A key finding of their research was that the maximum deflection caused by a moving load was approximately 2 to 3 times larger than the maximum deflection resulting from a static load at the midspan. Furthermore, they observed that the moment of occurrence of the maximum deflection differed from the static case. Specifically, as the velocity of the mass increased, the maximum deflection occurred at the midspan at a later moment.

III. BRIDGE AND VEHICLE SYSTEM

A. General equations of motion

In the field of vehicle-structure interaction, the vehicle and structure are described by their differential equations of motion 2. The subscript $b$ and $v$ refer respectively, to the bridge and vehicle system.

$$\begin{align*}
M_b \ddot{X}_b + C_b \dot{X}_b + K_b X_b &= F_b \\
M_v \ddot{X}_v + C_v \dot{X}_v + K_v X_v &= F_v
\end{align*} \quad (2)$$

With $M$, $C$ and $K$, respectively, the global mass, damping and stiffness matrices. The vectors $X, \dot{X}, \ddot{X}$ represent the displacements, velocity and acceleration of the system and $F$ is the force vector. Both systems are modelled and solved in the finite element software Abaqus.

B. Assumptions

To reduce the complexity of the wheel-rail interaction, some assumptions are made in general during the simulations. The primary objective of this study is to ascertain the overall behavior of the bridge structure itself under the passage of a train. However, it is important to note that the accuracy of the contact surfaces between the wheel and the bridge may be compromised due to the underlying assumptions made in the analysis.

1) The wheel and rail always have contact with each other.
2) The train moves at a constant speed over the bridge, so it is assumed that the train does not accelerate on the bridge.
3) Only the vertical wheel-rail interaction is considered, the lateral and longitudinal interactions are neglected.
4) No track irregularities are taken into account, the rails are assumed to be perfect.
5) The components of the vehicle are considered rigid, so the elastic deformation during vibration is neglected.
6) The track and the bridge are considered as one whole element, so there is no interaction between the track and the bridge.

C. Train model

A train model is a spring-damper-mass model or a combination of multiple spring-damper-masses. The development of train models is shown in figure 2.

D. Bridge model

The bridge model is a three-dimensional model established in the finite element software. The beam is divided into a finite number of cubic elements.

IV. Solution algorithms

A. Direct integration method

In the direct integration method (DIM), the train and bridge are in direct contact with each other. The equations of motion given in 2 are strongly coupled and solved in each time step. This means that for each time step, so each iteration step, both systems must converge. The number of time steps influences the accuracy of the results.

B. Intersystem iteration method

The intersystem iteration method (ISIM) divides the bridge and train model into two subsystems that are separately solved and coupled by the wheel-rail interaction forces. First, the train subsystem is excited by some random track irregularities defined by the German low disturb density spectrum [3]. These track irregularities excite the train only in the vertical directions and give resulting reaction forces in the wheels. Subsequently, these reaction forces are extracted from the train subsystem and subjected to the bridge subsystem as moving point loads. In the next step, the deflections of the bridge are measured in the bridge subsystem and stored. Because it is considered that the wheels and bridge are always in contact with each other, these deformations of the bridge are given as new boundary conditions of the vehicle. A new iteration loop is started with the vertical deflection of the bridge instead of the random track irregularities. This iteration loop goes further until convergence is reached. So the difference between the wheel-rail interaction forces calculated in step i and step i – 1 must be smaller than the predefined convergence check of 10 N.

V. Validation of the models

A. Case A: simply supported beam with moving spring-mass

The first case that is considered is shown in figure 4. The mass of the wheel and damping is neglected, therefore the

---

Fig. 1: Vertical deflection of a simply supported beam under a moving mass at different velocities, measured from the experiments of Willis and Stokes [2].

Fig. 2: Evolution of the vehicle model in time
complexity of the motion equations (2) of the bridge and train is significantly reduced. This allows us to find an analytical solution to the considered wheel-rail interaction model. The properties considered for the beam and the spring-damper are given in table 1.

The analytical simulations are based on the Euler-Bernoulli beam theory, while the numerical simulations in Abaqus are modelled in 3D and consider the Timoshenko beam theory. The shear forces are neglected in the Euler-Bernoulli beam, which results in higher beam stiffnesses compared with the Timoshenko beam theory that includes shear forces. Abdar-
B. Case B: simply supported damped beam with series of moving spring-damper-mass models

Once the behavior of a real train is simulated, the damping cannot be further neglected. Arvidsson et al. [5] investigate the response of a simply supported beam under the influence of the Swedish Green train. In order to validate the intersystem iteration method (ISIM) and the direct integration method (DIM) proposed in this dissertation, the same model as presented in Arvidsson et al.’s paper [5] is recreated and subjected to a comparative analysis. The loading pattern of the Swedish train is given in figure 6. The properties for the beam and spring-damper-mass model considered in this case are also given in table 1. Rayleigh-damping is considered to define the damping of the beam. The damping ratio of the beam is constant and the Rayleigh parameters are defined by equation 3. Therefore, the first and second bending frequencies of the beam are determined in Abaqus, which comes out in \( f_1 = 3.86 \text{ Hz} \) and \( f_2 = 15.76 \text{ Hz} \). The resulting Rayleigh parameters are \( \alpha_R = 0.03059 \) and \( \beta_R = 0.0005372 \).

\[
\xi_i = \frac{\alpha_R}{2f_i} + \frac{\beta_R f_i}{2}
\]  

The spring-damper-mass models will pass the beam at resonance speed, which is influenced by the first bending frequency \( f_1 \) of the beam and the carriage length \( L_c = 26.6 \text{ m} \).

\[
V_{res} = f_1 L_c
\]

The vertical deformation at midspan of the beam due to the passage of the Swedish Green train at resonance speed is given in figure 7. The horizontal axis is normalized by the time the last wheel leaves the beam, \( t = 2.37 \text{ s} \). The frequency of the different simulations is the same, but the amplitudes differ significantly. As explained, the beam and vehicle model are directly coupled with each other in the DIM, meaning that the motion of the first wheel has a direct influence on the motion of the second wheel. So, a small error could give a cascading effect and should explain the increasing error between the DIM and the ISIM during the passage of the train.

C. Case C: simply supported damped beam with series of train-bridge dynamic interaction models

The train model considered in this paper is the spring-damper-mass model, where the different wheels are not connected with each other by a bogie. Figure 3 compares already the difference between the four different vehicle models in function of the velocity. The choice of train model becomes important when considering a train that moves at resonance speed. Therefore, both methods considered in this paper are compared with the results of a train-bridge dynamic interaction model. The handbook 'Bridge vibration and controls' of Xia et al [6] presents the vertical deformation of a simply supported beam under the Italian high-speed train ETR500Y modelled as the train-bridge dynamic interaction model (TBDIM). The loading pattern of the Italian high-speed train ETR500Y is given in figure 8 and the considered properties of the train and beam are given in table 1. The damping of the beam is also defined by Rayleigh damping. To compare the spring-damper-mass model with the train-bridge dynamic interaction model, the stiffness of the spring-damper-mass model is adapted so that the bogie frequency \( f_{vb} \) of the two vehicle models is the same. The bogie frequency is given by equation 5, where \( K_V \) and \( K_{VV} \) are the vertical spring stiffness of respectively the primary suspension and secondary suspension. The bogie frequency of the train-bridge dynamic interaction model is \( f_{vb} = 4.8 \text{ Hz} \), which results in an equivalent spring stiffness \( K_V = 404370 \text{ kN/m} \) for the spring-damper-mass model.

\[
f_{vb} = \frac{1}{2\pi} \sqrt{\frac{2K_V + K_{VV}}{M_i}}
\]

The resulting vertical deformations at midspan of the beam during the passage of the Italian high-speed train at resonance speed are given in figure 9. The accuracy of both methods compared with the simulation from the paper of the train-bridge dynamic interaction model are given in table 2. The amplitudes simulated by the DIM give again a quite large overestimation of the vertical deformation.

D. Efficiency of the methods

Finally, the efficiency of the direct integration method (DIM) and intersystem iteration method (ISIM) are compared in table 2. The calculation time of each simulation is summarized. The total duration of the ISIM significantly exceeds...
that of the DIM. To compare the accuracy over the different cases, the relative root mean square error is determined. The predicted values of both methods are compared with the values based on the papers. For the different cases, the error of the ISIM is 1.6 to 1.7 times smaller than the error of the DIM. As the complexity of the different cases increases, the error becomes more pronounced. The accuracy of both simulation methods increases significantly for case C and becomes unacceptable for the DIM. The DIM is not immediately discarded because it was already given in figure 3 that the way of modelling gives very different results at resonance speed. So, the different cases should be recalculated for lower train velocities to give a better view of the correctness of both methods.

Table 2: Comparison of the direct integration method (DIM) and the intersystem iteration method (ISIM) in calculation time and accuracy for the different simulations

<table>
<thead>
<tr>
<th>Method</th>
<th>Calculation time [s]</th>
<th>RRMSE [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
<td>DIM</td>
<td>349</td>
</tr>
<tr>
<td></td>
<td>ISIM</td>
<td>2160</td>
</tr>
<tr>
<td>Case B</td>
<td>DIM</td>
<td>740</td>
</tr>
<tr>
<td></td>
<td>ISIM</td>
<td>44100</td>
</tr>
<tr>
<td>Case C</td>
<td>DIM</td>
<td>583</td>
</tr>
<tr>
<td></td>
<td>ISIM</td>
<td>33600</td>
</tr>
</tbody>
</table>

VI. SIMULATION AT NON-CRITICAL SPEED FOR CASE B

As in reality it is avoided to pass the railway bridges at their resonance speed, it is important to compare the methods at the applicable working area they are used for. The simulation of case B is recalculated for both methods at a lower train speed of 15 m/s and the results are given in figure 10. The global vertical deformation of both methods is the same, but the influence of the different vehicles on each other is noticeable again in the DIM. As the train is moving at a much lower speed than the resonance speed, the beam would not vibrate anymore around the first bending frequency of the beam. The response of the bridge at other speeds than the critical is priorly determined by the driving frequency $f_{dr}$ and the dominant frequencies $f_{d,n}$ given by equation 6 and 7, respectively [7].

$$f_{dr} = \frac{nV}{2L_c} \quad (6)$$

$$f_{d,n} = \frac{nV}{L_b} \quad (7)$$

Where $n$ is the considered mode, $V$ is the train speed and $L_c$ is the length of the carriage. The length $L_b$ is determined so that $L_b/V$ represents the total duration of the passage of the train. The driving frequency $f_{dr} = 0.0308$ is very low and results in the total time of the vertical deformation during the passage of the train. The higher frequencies that occur in the response of the beam are due to the dominant frequencies. The frequency response of the vertical deformation for case B at non-critical speed is given in fig 11 and shows the influence of the different modes. The first mode ($n = 1$) is the most predominant for both methods and gives the overall up and down movement of the beam during the passage of the train.

In the graph of the DIM, an additional higher frequency is remarkably defined by the seventh mode. The relative root mean square error between both simulations is only 6.9 %, which is significantly reduced compared with the simulations at resonance speed (16 – 25 %). Both, the direct integration method and the intersystem iteration method, result in very similar results at non-critical speeds and are recommended for further use to simulate the dynamic wheel-rail interaction.

VII. CONCLUSION

Finally, it can be concluded that the direct integration method and the intersystem iteration method are two useful methods. The direct integration method overestimates the vertical deformations significantly but is preferred to have a first attempt at the behavior of a railway bridge under the passage of a train. Although the intersystem iteration method is more computationally intensive, it gives more accurate results to really predict the structural health of the railway bridge.

VIII. FUTURE WORK

As this paper focuses solely on the vertical wheel-rail interaction forces, it is of interest to explore the influence of the horizontal wheel-rail interaction. This investigation
would contribute to a more comprehensive understanding of the overall wheel-rail interaction. It is important to assess whether incorporating the horizontal forces would lead to more accurate results without a significant increase in computational time.

Furthermore, to enhance the reliability and applicability of the developed methods for analyzing the behavior of railway bridges under train loading, it is recommended to validate these models using real strain measurements obtained from railway bridges. This validation process would not only verify the accuracy of the numerical models but also provide valuable insights into potential shortcomings or deficiencies in the methods. This additional information would greatly contribute to improving the models and ensuring their effectiveness in practical applications.

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<th>Description</th>
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<tbody>
<tr>
<td>DAF</td>
<td>Dynamic amplification factor</td>
</tr>
<tr>
<td>DIM</td>
<td>Direct integration method</td>
</tr>
<tr>
<td>DOF</td>
<td>Degree of freedom</td>
</tr>
<tr>
<td>FEM</td>
<td>Finite element method</td>
</tr>
<tr>
<td>HSA</td>
<td>Hybrid solution algorithm</td>
</tr>
<tr>
<td>ISIM</td>
<td>Intersystem iteration method</td>
</tr>
<tr>
<td>LCM</td>
<td>Loosely coupled method</td>
</tr>
<tr>
<td>MCFM</td>
<td>Moving constant force model</td>
</tr>
<tr>
<td>MHFM</td>
<td>Moving harmonic force model</td>
</tr>
<tr>
<td>MMM</td>
<td>Moving mass model</td>
</tr>
<tr>
<td>MSDMM</td>
<td>moving spring-damper-mass model</td>
</tr>
<tr>
<td>PSD</td>
<td>Power spectral density function</td>
</tr>
<tr>
<td>RMSE</td>
<td>Root mean square error</td>
</tr>
<tr>
<td>RRMSE</td>
<td>Relative root mean square error</td>
</tr>
<tr>
<td>RF</td>
<td>Reaction force</td>
</tr>
<tr>
<td>SCM</td>
<td>Strongly coupled method</td>
</tr>
<tr>
<td>TAVBM</td>
<td>Two-axle vehicle bridge model</td>
</tr>
<tr>
<td>TBDIM</td>
<td>Train-bridge dynamic interaction model</td>
</tr>
<tr>
<td>TTBDIM</td>
<td>Train-track-bridge dynamic interaction model</td>
</tr>
<tr>
<td>TSI</td>
<td>Timestep iteration method</td>
</tr>
<tr>
<td>3D</td>
<td>Three-dimensional</td>
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Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\Delta t$</td>
<td>Time step</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Mass density</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Damping ratio</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Circular frequency</td>
</tr>
<tr>
<td>$A$</td>
<td>Area of the cross-section</td>
</tr>
<tr>
<td>$C_V$</td>
<td>Vertical damping coefficient of the primary suspension</td>
</tr>
<tr>
<td>$C_{VV}$</td>
<td>Vertical damping coefficient of the secondary suspension</td>
</tr>
<tr>
<td>$E$</td>
<td>Elasticity modulus</td>
</tr>
<tr>
<td>$f_d$</td>
<td>Dominant frequency</td>
</tr>
<tr>
<td>$f_{dr}$</td>
<td>Driving frequency</td>
</tr>
<tr>
<td>$f_i$</td>
<td>$i$th bending frequency</td>
</tr>
<tr>
<td>$f_{vb}$</td>
<td>Bogie frequency</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational constant</td>
</tr>
<tr>
<td>$I$</td>
<td>Bending stiffness</td>
</tr>
<tr>
<td>$K_V$</td>
<td>Vertical spring stiffness of the primary suspension</td>
</tr>
<tr>
<td>$K_{VV}$</td>
<td>Vertical spring stiffness of the secondary suspension</td>
</tr>
<tr>
<td>$L$</td>
<td>Length of the beam</td>
</tr>
<tr>
<td>$L_c$</td>
<td>Carriage length</td>
</tr>
<tr>
<td>$L_{inter}$</td>
<td>Axis distance between two successive wheels</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass per unit length of the beam</td>
</tr>
<tr>
<td>$M_1$</td>
<td>Mass of the bogie</td>
</tr>
<tr>
<td>$M_2$</td>
<td>Mass of the car body</td>
</tr>
<tr>
<td>$M_w$</td>
<td>Mass of the wheel</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
</tr>
</tbody>
</table>
List of Symbols

\( V \) Velocity
\( V_{res} \) Resonance velocity
\( y(x, t) \) Vertical deflection of the beam in function of time and space
\( Z(t) \) Displacement of the bogie

**Subscript**

\( v \) Vehicle
\( b \) Bridge
\( dof \) Degree of freedom
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Chapter 1

Introduction

Nowadays, sustainability becomes more important and affects the research topics in infrastructure. The demolishing of existing bridges increases the waste that should be destroyed, which has a huge impact on the environment. The usefulness of recycling building materials into new constructions increases and studies can already be founded in several papers [19] [20]. By monitoring the structural health of bridges, it is possible to extend the life time of existing bridges by strengthening or repairing the weak areas on time. A large European project called 'Sustainable bridges' [21] aims to have a more sustainable society. Therefore, a modal shift in the use of transport is needed. The goal is to stimulate people to take public transport or bicycle instead of the car, as decreasing the number of cars on the road creates less pollution and has a large positive effect on the environment. To realize this sustainable modal shift, public transport options must become more attractive. The paper of Gaudry [1] discussed the evolution of transport use in France. Figure 1.1 shows the distance in kilometers per person per day traveled for each type of transport for data from 1800 until 1990. Due to the large increase in the use of cars and buses, the train is no longer the main transport from 1900 onwards. This results in a huge impact on the combustion of exhaust gases and an increase in traffic density on roads. Therefore, clarifies the importance of the modal shift to achieve a more sustainable manner of transportation.

![Figure 1.1: Travelled distance in kilometers per person per day in France for the different transport options [1]](image)

To have more attractive public transport, the European railway network should be enlarged and different adaptations should be taken. The 'Sustainable Bridges' project focuses on the influence of the modal shift on railway bridges. To increase the capacity of railway bridges, there are three important goals to achieve: (1) Increasing the transport capacity, (2) Increasing of the residual life time of the
bridges, (3) Improving the structural health monitoring and repair systems. Increasing the transport capacities results in an increase in the axle loads of train vehicles. As it is not possible to make each existing railway bridge wider to provide more rails, the fastest solution will be to increase the axle loads on the bridges. Due to the higher loads acting on the bridge, the end-of-life span could decrease significantly. To guarantee the safety of the design, measurements and predictions of the railway bridges must be performed to have knowledge about the behavior of the railway bridges.

The European project 'Sustainable bridges’ summarized in primer work \[2\] the demography of the European Railway bridges. Graph 1.2 summarizes the age of steel bridges measured in different countries in Europe: Austria, Belgium, Czech Republic, Denmark, Ireland, France, Germany, Hungary, Italy, Poland, Portugal, Slovakia, Spain, Sweden, Switzerland and the UK. This illustrates that in 2004 already 39.7\% of the steel railway bridges were older than 50 years and 27.9\% were older than 100 years. Being aware that a large number of steel railway bridges reach their end-of-life span increases the importance of monitoring the structural health of these railway bridges.

![Age of metal bridges in 2004](image)

Figure 1.2: Age of metal bridges measured in different countries in Europe in 2004 \[2\]

The dynamic behavior of the railway bridges during the passage of a train becomes interesting. So, in this master dissertation, a train-bridge interaction model is created to simulate the deformations of the bridge in critical sections under a moving train. Therefore, the software Abaqus is used to make numerical models of the wheel-rail interaction between the train and bridge.

1.1 The Temse Bridge

For this master dissertation, a specific case study of the Temse Bridge is considered. The bridge is located in Temse and crosses the Scheldt River. The first construction of the Temse Bridge dates back to 30 November 1870. It was a railway bridge designed by Gustave Eiffel. During the First World War, the bridge was totally destructed. As the increased use of cars made it important to have a road bridge, the reconstruction of the Temse Bridge started in 1949 and finished six years later. Nowadays, a second Temse bridge is built near the old one, to allow a larger amount of traffic. For this dissertation, only the movable part (the part that opens/closes for the passage of boats) of the railway bridge is considered, which is known as the weakest part of the bridge. Strain measurements were performed to study the behavior of the bridge while passing a train.
In previous years, the finite element model of the Temse bridge was modeled and validated to strain measurements on the real structure. In this model, only a quasi-static load case had been used and the dynamic effects were considered with a dynamic amplification factor. The master dissertation of Kazimir [22] already implemented dynamic analysis with a vehicle-structure interaction model. A numerical integration method in the time domain was used to solve the dynamic system. For simple cases, this proves a better simulation than the static one as it corresponds better with experimental measurements. When applying the method to the case study bridge, the model has a large number of degrees of freedom, causing it to be extremely computationally expensive. To limit the time needed to solve this model, less time-consuming methods are required.
Chapter 2

Literature review

In this literature review, an overview of the evolution in modelling train-bridge dynamic interaction will be given. In section 2.1, the importance of moving loads and the use of modal decomposition for modelling train and bridge dynamics will be discussed.

In addition to further research on the influence of moving loads on the dynamic behavior of bridges, attention is also drawn to the development of increasingly sophisticated train models. The evolution of different train models and their influence on the dynamic behavior of bridges is described in section 2.3.

In order to model the dynamic interaction between a train and bridge, assumptions need to be made about the mechanical behavior at the interface of both systems. These assumptions are typically incorporated in wheel-rail interaction models. Section 2.4 provides an overview of wheel-rail interaction models used in literature.

Lastly, section 2.5 describes different integration methods utilized for solving the bridge-train system.

2.1 Importance of including dynamic loads

The urge to understand the dynamic behavior of railway bridges dates back to the collapse of the Chester Bridge in the 19th century. This event was the motivation for Willis [4] and Stokes [23] to study the impact of a passing vehicle on the dynamic behavior of a bridge. Willis presented an empirical equation that aims to describe the deflection of a bridge. The equation was based on experimentally measured displacements of a beam under a moving mass shown in figure 2.1, which was later named the moving mass model (MMM). Willis’ equation neglects the centrifugal force and assumes the same deformation path as the static force. The goal of Stokes’ work was to determine the influence of the vehicle’s speed on the deflection of the bridge. Stokes presented a mathematical description that does account for the centrifugal force and describes a deformation path similar to that measured in the experiments of Willis given in figure 2.2. This graph shows the measured vertical deflection of the test beam at midspan for different values of $\beta$ given by equation 2.1, where $2a$ is the length of the beam and $S$ the static deflection at midspan. The static equilibrium curve gives the deformation at midspan due to a static point load. The larger the value of $\beta$, the lower the velocity and the more the deformation curves of the moving load approaches the static equilibrium curve. By an increasing value of $\beta$, so a higher velocity, the moment in time when the maximum deformation occurs at midspan shifts to the support.

$$\beta = \frac{ga^2}{4V^2S} \quad (2.1)$$

Stokes concluded that the deflections due to the dynamic loading are larger than the peak deflection at the center of a bridge caused by an equivalent static point load at the center. The works of Stokes and
Willis sparked the interest of the research community towards developing a complete mathematical description for the bridge deflection under a moving load.

Figure 2.1: Willis’s device for dynamic deflection testing of beams [3]

Zimmerman [24] further developed the problem of the impact on the deformation of a bridge of moving loads in 1896. All studies up to this point in time neglected the mass of the beam compared to the mass of the single moving load. Krylov [25] was the first to consider the opposite. He assumed that the mass of the moving load is negligible compared to the mass of the beam and considered the moving constant force model (MCFM). Timoshenko [26] and Inglis [27] only considered moving loads with a constant velocity, while Lowan [28] was the first to consider moving loads with variable speed by including a function of time into the equation that describes the deformation of a beam.

During the 20th century, the dynamic behavior of bridges was not only being researched in England, but also in Germany. As mentioned before, Zimmerman extended the works of Willis and Stokes. During the period spanning the First and Second World Wars, poor communication of the new findings hampered progress and resulted in a considerable amount of redundant research being done. Steuding (1934) [29] and Schallemcamp (1937) [30] studied bridge vibrations under moving loads, considering both the mass of the vehicle and the mass of the beam.

In 1972, Fryba [8] gained access to data of strain measurements of railway structures measured during field tests all over the world, and caused by different types of vehicles. He used this to assemble knowledge on the effect of moving loads on bridge dynamics. This work already covered multi-axle vehicle load cases. It had long become clear that the consideration of dynamic loads for determining bridge deformations and the resulting stresses is important. Stokes had already mentioned that the deflections induced by dynamic loads are larger than those induced by an equivalent static load. In a later work of Timoshenko [3], he estimated that the dynamic deflections are approximately 1.64 times larger than the deflection under the same static load. Where, the factor 1.64 depends on the speed of the vehicle and formed the basis for what is known today as the dynamic amplification factor (DAF) that is used to account for dynamics in many design practices (e.g. NBN EN 1991-2 [31], ASTM-E1318-09 [32]). The dynamic amplification factor is in more detail discussed by Patrick [33], which concludes that attention is needed when using the DAF. The DAF is an empirical constant, that depends on several properties such as the span length, natural frequency of the bridge, the weight, etc., which is not necessarily conservative. Furthermore, for fatigue life assessment, accurate knowledge of the
stress ranges is imperative as the influence of a small discrepancy in the stress range has a significant influence on the fatigue life due to the log-log scale of the S-N curve. Hence, high-fidelity modelling of train-bridge interaction is important.

In 1996, Fryba [34] summarized the science of railway bridge dynamics in a book that serves as a handbook for civil engineers. Special attention is paid to his own theoretical contribution applied to railway bridges. In addition, the influence of different parameters on the dynamic behavior of bridge and train is analyzed. He also included the vertical and lateral forces acting on the bridge and train.

Several scientists performed field measurements on railway bridges for further developments of the theoretical vehicle-bridge interaction model. The calculated results from the theoretical models are compared to the field measurements to approximate the reality as well as possible (e.g. [35]). More complex and detailed mathematical equations are formed to simulate the motion of the vehicle and bridge.

2.2 Modal decomposition method

The need for less computationally expensive methods has led to the development of the modal decomposition method. Adomian [36] introduced the Adomian modal decomposition method and applied it to physical problems like the N-body problem, which wants to solve the motion of N bodies that interact with each other. This method is used to solve non-linear equations; for example differential equations such as the equation of motion of dynamic structures. Cherruault [37] proved convergence to an exact solution of the non-linear equation by applying the Adomian modal decomposition method.

In the field of vehicle-structure interaction, both structure and vehicle can be modelled as an elastic subsystem, described by their own differential equation of motion. The modal decomposition method can be used to solve both equations separately.

For the case of train-bridge interaction, the differential equations of motion of the bridge and train systems are generally expressed as:

\[
\begin{align*}
M_b \dddot{X}_b + C_b \ddot{X}_b + K_b X_b &= F_b \\
M_v \dddot{X}_v + C_v \ddot{X}_v + K_v X_v &= F_v
\end{align*}
\] (2.2)

With \(M, C, K\), respectively, the global mass, damping and stiffness matrices. The vectors \(X, \dot{X}, \ddot{X}\) represent the displacements, velocity and acceleration of the system and \(F\) is the force vector.

The equations of motion of each subsystem are coupled, which means that different variables are given by more than one equation. One could use a direct integration method to solve this system of equations in the time domain, as was done in the preceding master dissertation of Kazimir [22], but this is a very time-consuming solution method. Therefore, it is of interest to decompose the system of \(N_{dof}\) coupled equations of motion into \(N_{dof}\) uncoupled equations of motion, where \(N_{dof}\) represent the number of degrees of freedom in the system. From each uncoupled equation, a corresponding mode shape can be obtained which represents the motion of the structure corresponding to the \(n^{th}\) natural frequency of the structure. The total dynamic response of the subsystem is the superposition of all mode shapes.

The property of the orthogonality of modes makes it possible to decouple the equations of motion. The coupled system of equations is simplified to a superposition of \(n\) independent modal equations as shown in equation 2.3.

\[
\begin{align*}
\dddot{Q}_b + 2 \xi \omega \dot{Q}_b + \omega^2 Q_b &= \Phi^2 F_b \\
\dddot{Q}_v + 2 \xi \omega \dot{Q}_v + \omega^2 Q_v &= \Phi^2 F_v
\end{align*}
\] (2.3)

where the diagonal matrices \(\omega, \xi\) represent the circular frequency and damping ratio, respectively. The vectors \(Q, \dot{Q}, \ddot{Q}\) represent the displacements, velocity and acceleration in a generalized coordinates
system and can be rewritten in the global coordinates system with the normalized modal matrix $\Phi$. This matrix consists of the mass-normalized eigenvectors of the subsystem given by equation 2.4.

$$\begin{cases} Q_b = \Phi_b^2 X_b \\ Q_v = \Phi_v^2 X_v \end{cases}$$

(2.4)

This method not only simplifies the solution by dividing the coupled equation in $N_{dof}$ decoupled equations that can be solved separately. It is also well known that not all existing mode shapes are excited to the same degree. Olsson [5] illustrates the effect of the number of included modes on the contact force $f_c(t)$ between the wheel and the bridge. Figure 2.3 gives the contact force normalized by the total weight of the vehicle in the function of the time $t$, which is normalized by the total duration $\tau$ of the vehicle that passes the bridge. The continuous line corresponds to the contact force ratio obtained when considering only a single mode, the dashed line corresponds to the contact force ratio obtained when considering five modes [5]. The figure shows a good agreement between both curves. Considering only one mode is sufficient to model the wheel-rail interaction. Higher modes have less influence and only the first few are generally relevant. Thus only a subset of the eigenmodes, more precisely the first $N_{eig}$ modes with the lowest eigenfrequencies need to be determined. As such, the computational effort can be significantly reduced.

![Figure 2.3: The effect on the contact force ratio of including different numbers of vibration modes: one mode (continuous line), five modes (dotted line) [5]](image)

Once the train and bridge subsystems have been solved separately, they need to be coupled again in order to account for the interaction effects. In the first step of modal decomposition, there is no interaction yet between the two subsystems. The coupling is achieved through a wheel-rail interaction model that describes the contact forces at the interface between the two systems. As mentioned, wheel-interaction models will be discussed in section 2.4. Evidently, to obtain the complete bridge deflection history for a moving load, an iterative calculation is required.
2.3 Modelling of the train subsystem

During the past two centuries, different train-bridge interaction models have been developed [6]. Mainly, the way of modelling the train system varies significantly across the different models. The evolution of the dynamic train-bridge models as a function of time is shown in figure 2.4.

Willis [4] and Stokes [23] already used a single moving load to show the significance of dynamic train bridge modelling. The history of the moving constant force model (MCFM) and the moving mass model (MMM) were discussed in the previous section 2.1. In the moving harmonic force model (MHFM), the sinusoidal force moves along the beam. In these three models, the dynamic effect of the vehicle on the bridge is neglected.

Biggs [7] introduced the moving spring damper mass model (MSDMM), which is the first model that includes the dynamic behavior of the vehicle. Here, the train is modelled by a moving mass on a spring-damper system. The latter represents the rail carriage’s suspension system. In his work, Biggs [7] compared the deflections at the midspan of the theoretic model with field measurements, as illustrated in figure 2.5. The vertical axis shows the dynamic deflection divided by the maximal static deflection whilst the horizontal axis shows the position of the vehicle load. The measurements of the deflection of the bridge were performed for a two-axle heavy truck passing over the bridge. The results of the moving spring damper mass model were found to be in good agreement with the experimentally obtained measurements.

---

Figure 2.4: History of the different train models [6]

Figure 2.5: Comparison of the deflection at midspan of a steel bridge obtained by a theoretical model MSDMM and actual field test data for a two-axle truck [7]
Fryba [8] compared various models with experimental measurement data. The time variation of the deflections was calculated with each model and compared to measurements obtained from controlled load tests on a railway bridge with a known locomotive, i.e. a two-cylinder steam locomotive. In figure 2.6 the different models from his work are compared with each other and the actual experiments. The deflection at the center of the beam with time is shown for two cases: the moving harmonic force model (MHFM) in blue and the moving spring-damper-mass model (MSDMM) in orange. The calculations were performed for different velocities ($V = 34.7/40.6/46.5 km/h$) of the locomotive. For a speed of 34.7 km/h, the relative root mean square error (RRMSE) between both methods is only 0.32/%, so a good agreement between both models was found. For the two larger velocities in cases b and c, the difference in error between the predictions by MSDMM and the MHFM becomes 2%. This gives still a good agreement between both methods for the small increase in velocity. In both the MSDMM and MHFM predictions, the maximum values do not occur at the same moment as in the experiments and give a shift in time. When only looking at the value of the amplitude for the deflection, the MSDMM agrees best with the experiment. The error made by the MSDMM is $1 - 2\%$ lower than by the MHFM and gives a slightly more accurate model. This wouldn’t exclude the importance of including the dynamics of the vehicle in determining the bridge deflection.
Figure 2.6: The vertical deflection of the bridge at midspan for a passing steam locomotive at different speeds: comparison of experimental values with predictions based on MHF and MSDM models [8]

Fryba [8] extended the MSDM vehicle model to a two-axle vehicle bridge model (TAVBM) through the addition of an extra spring-damper-mass component. In his work, he measured the stresses at the center of the bridge as a function of time. These experiments were performed on the Paar bridge in Germany loaded with an electrical locomotive E03. Figure 2.7 compares the stresses calculated from the two-axle vehicle bridge model to the measurement data. The theoretic model and the experiment are in good agreement.
In general, a vehicle model comprises a car-body, two bogies and four wheelsets connected with a two-layer suspension system, namely the upper and lower suspension or respectively the secondary and primary suspension. The upper suspension comprises the spring and dashpot between the vehicle body and the bogie. The lower suspension is comprised of the spring-dashpot system connecting the bogie and the wheelset and is further discussed in section 2.4. Diana [9] developed such a train model, called the train bridge dynamic interaction model (TBDIM). This model has 23 degrees of freedom, where the car-body, the bogies and the wheelset are assumed to be rigid body. Diana [9] compared his model with experiments on a steel frame railway bridge crossing the Po River. Figure 2.8 shows the results hereof; the analytical model has a good agreement with the measurements. The analytical model resembles a regression curve that neglects the small fluctuations in deflection but gives a good approximation of the global beam deflection due to the load on the bridge.

Zhai and Sun [38] compared the vertical wheel-rail interaction forces for three models: MSDMM, TAVBM and TBDIM. They concluded that the wheel-rail interaction forces for MSDMM and TAVBM are respectively 13% and 9% smaller than those obtained with TBDIM. This means that the deflections under the MSDMM and the TAVBM are underestimated compared with the TBDIM.

Because of the increasing complexity of train models presented in research papers, it becomes increasingly important to consider the balance between model complexity and accuracy gain with respect to the desired results. Arvidsson and Karoumi [10] compared the dynamic response of a simply supported beam for different train models. Specifically, they compared the accelerations at the midspan of the beam determined with the MCFM, MSDM, TAVBM and TBDIM for two span lengths: 6 m and 36 m. They considered the resonance train speeds that are equal to 320 $km/h$ and 370 $km/h$ for
the short and long beams, respectively. Figure 2.9 shows their results. The results for the MSDM, TAVVB and TBDI models are very close to each other for a span length of 6 m. For the span length of 36 m, larger discrepancies occur at resonance speed because the first eigenfrequency of the beam is close to that of the bogies (4 – 6 Hz). The choice of train model becomes important at resonance speeds, especially if the first bending frequency of the beam is close to the bogie frequency. The same conclusion was also found in the experiment of Arvidsson et al. [10], where they compared the MCFM, MSDM and the TAVBM for different span lengths of the beam. The MCF model shows significant discrepancies from the other results, especially for a short beam. The amplitude of the MSDM and TBDIM are respectively 19% and 8% lower than those simulated by the TAVBM. For train speeds out of the range of resonance, the difference in amplitude between the three models becomes significantly smaller. Based on these findings, a reasonable conclusion can be drawn that the MSDM, TAVBDIM, and TBDIM models exhibit highly comparable results for the vertical acceleration of the beam when the train speeds are significantly distant from the resonance speed.

![Comparison of accelerations for different train models](image)

Figure 2.9: Comparison of the accelerations of a simply supported beam obtained for different train models (MCF, MSD, TAVB and TBDI)[10]

When the dynamic behavior of the vehicle becomes important, there are significant differences between the results obtained from the four compared models. The MCFM model is not capable of accounting for the influence of the vehicle. When there is interest in running comfort for passengers of the vehicle, more sophisticated models than the MCFM model are needed.

The dynamic model of Diana was further extended by Zhai et al. [39] through the inclusion of the train-
track-bridge coupling. The model is appropriately called the train-track-bridge dynamic interaction model (TTBDIM). In this model, the tracks and bridge are no longer considered as a single system, but as two subsystems. The model was validated with experimental measurements originating from the Yellow River bridge [40].

Cheng and Cheung [11], showed that the influence of the vibration of the track subsystem has no significant influence on the dynamic response of the bridge. Cheng et al. [11] performed experiments on a prestressed concrete bridge. Three different models were considered: the discrete model with a spring-damper system at a fixed distance, the continuous model with a continuous spring-damper support and the model where the track structure is ignored (TBDIM). The dynamic amplification factor for deflections at midspan $D_d$ for a simply supported single-span bridge was determined with the three considered models. The comparison is shown in figure 2.10. The work of Cheng and Cheung [11] is obviously not a comparison for a steel bridge, but it already indicates that including the vibration of the track subsystem as proposed in the TTBDI model does not considerably influence the deflections during the passage of a train.

![Figure 2.10: Influence of the track structure on the dynamic magnification factor for deflections at mid-span for a simply supported single-span concrete bridge [11].](image)

To this day, new models continue to be developed that aim to simulate the dynamic behavior of bridges due to moving trains. Li et al. [41] used a Bayesian deep learning approach to simulate the random vibration of bridges due to the dynamic interaction with a vehicle. The method was applied to a real railway girder bridge and results were compared with the results from the vehicle bridge dynamic interaction model. Both modelling techniques had a good agreement, but the Bayesian deep learning approach needs further investigation.

### 2.4 Wheel-rail interaction

To simplify the calculations, the whole bridge-train system is divided into two subsystems that are decoupled and separately solved, for example by modal decomposition as explained in section 2.2. A wheel-rail interaction model is needed to describe how both subsystems influence each other at the interface. Each wheel-rail interaction model approximates the reality in a different way. In general, wheel-rail interaction models describe the contact forces that act between the rail and wheels at their interface.
2.4.1 Vertical interaction

The wheel-rail vertical interaction is the first group of models that is limited to normal force transfer between wheel and track. Within this category, both linear and non-linear models exist as example the corresponding assumption and Hertz contact theory, respectively. Zakeri and Tajalli [42] discussed in their work the validity of the linear model and the improved results using non-linear models. They concluded that non-linear models are necessary when considering track irregularities. Using a linear model with significant track discontinuities would considerably underestimate the wheel-rail interaction force. Within this context, an example of a linear and non-linear model is further discussed.

In the corresponding assumption, it is assumed that the wheels of the vehicle model and the track of the bridge model always have contact. Then the contact forces are a function of the relative displacements at the contact points, indicating this modelling approach gives no exact local detail of the stresses in the rails and wheels at the wheel-rail contact. As such this is a rather simple way to model contact between the track and wheel. Different works [35] [43] [44] use this method to solve the dynamic behavior of the bridge-train model. The vertical wheel-rail interaction forces consist of the interaction force between the bogie and wheelset, the inertia force of the wheelset and the static axle load.

In 1881, Hertz [45] presented Hertz’s law which gives the radius of contact between a sphere of elastic material and a surface in terms of the sphere’s radius, the normal force exerted on the sphere, and Young’s modulus for the material of the sphere. Hertz neglected adhesion between both surfaces and assumed two perfectly smooth surfaces, so no friction occurs. Yan and Fischer [46] applied the Hertz contact theory in the modelling of the wheel-rail interaction. They stated two disadvantageous of the Hertz theory. As Hertz assumed, friction between both surfaces is neglected. Secondly, the dimension of the Hertz contact area must be small in comparison with the dimensions of the elements in contact. Further, they compared the Hertz contact theory with the linear-elastic theory for different rail types. They concluded that the Hertz contact theory and the linear-elastic theory have a good agreement for modelling the contact pressure between the track and train, but only for a straight track.

2.4.2 Lateral interaction

The wheel-rail lateral interaction is the second group of models. These consider the tangent interaction forces. Similar to the vertical interaction, linear and non-linear models exist. Examples are the hunting assumption and the Kalker creep theory, respectively.

The hunting movement of the wheelset is given by the relative displacement between the wheels and track and excites the train-bridge model in the lateral direction, like the track irregularities. The hunting movement in the lateral direction can be described by a sinusoidal function. Due to the relative displacements between the wheels and the track, lateral interaction forces can be obtained. This method is similar to the vertical corresponding assumption, as the lateral interaction forces are a function of the relative displacement of the wheel and track. [35]

In 1967, Kalker [12] described the contact of two rolling devices rotating about their axis that are pressed together with the force $N$, this is illustrated in figure 2.11. Creep occurs due to the different circumferential velocities ($V^{-}$ and $V^{+}$) of the two elements when a force (the creep force) is transmitted to one of the two elements. This results in a contact zone with normal pressure distribution that results in creep of the elements. Spin occurs when the velocities ($V_{r}^{+}$ and $V_{r}^{-}$) of the elements around the axis perpendicular to the contact area are not similar. In Kalker’s dissertation, he describes the tangential forces $F_{x}$ and $F_{y}$ exerted through the elements at the contact surface.
Literature review

The Kalker linear creep theory is limited to the case where very small creep and small spin occur. By increasing the creep force, slip cannot be neglected and the Coulomb friction limit is reached. Later, Kalker [47] simplified his model to the Simplified linear Kalker theory and implemented it in a software package FASTSIM [48]. The wheel and rail are simplified by modelling as a spring system, where each point moves independently of the other. Finally, Kalker [49] developed a complete exact 3D Kalker creep contact theory that was implemented in the software package CONTACT.

Shen et al. [50] further optimized the Kalker creep theory. They concluded that the Kalker creep theory differs more from the exact contact modelling technique by increasing the spin. Shen et al. [50] multiplied the Kalker linear creep forces by a correction coefficient so that the theory is applicable for large creep and spin, called the Shen correction.

Meymand et al. [13] compared the different wheel-rail contact models. Figure 2.12 compares the lateral creep force for the previously mentioned models for two cases of spin creep ($\phi = 0.005$ and $\phi = 0.01$). The exact 3D model of Kalker (CONTACT) differs from the other. For smaller spin creep, the predicted creep force is larger than the Shen and FASTSIM models, while for larger spin creep the reverse hold. The results obtained by applying the Shen correction and the FASTSIM model are similar.

Figure 2.11: Illustration of the contact area of two rolling elements [12]

(a) Spin creep $\phi = 0.005$
(b) Spin creep $\phi = 0.01$

Figure 2.12: Comparison of the normalized lateral creep force for the models: FASTSIM, Shen and CONTACT. [13]

Salcher and Adam [51] combined in their work the corresponding assumption method for the normal
forces with the simplified Linear Kalker creep theory to couple the train model and bridge model. They assumed a series of MSDM models and a simply supported single-span steel railway bridge. Further, they determined the importance of including track irregularities in the interaction method to have knowledge about the dynamic response of the bridge and train while passing a moving vehicle. They concluded that the track irregularities have a significant impact on the vertical acceleration of the bridge and train, so the lateral wheel-rail interaction can’t be neglected.

2.4.3 Longitudinal interaction

Wheel-rail interactions also occur in the direction of travel. Zhao et al. [52] conducted a study where they subjected a wheelset to vertical, lateral, and longitudinal high-frequency vibrations to assess their effects on the vertical ($F_z$), lateral ($F_y$), and longitudinal ($F_x$) interaction forces. They concluded that significant longitudinal interaction forces only occur when the wheelset experiences rotational vibrations. These vibrations arise when there is a variation in torque around the wheel-axis, such as during the acceleration of the vehicle. When assuming a constant vehicle speed, the longitudinal interactions are considered negligible.
2.5 Solution algorithm

Different algorithms exist to solve the train-bridge coupling system, which all differ in the way the coupling relationship between the train and bridge subsystem is interpreted. The different methods only differ in approaching the problem in a mathematical way, but the physical interpretation of both methods should be the same. There are two main categories to solve the train-bridge system: the strongly coupled method (SCM) and the loosely coupled method (LCM). The former takes the bridge and train model as one integrated model with internal forces acting at the wheel-track interface. The latter solves the bridge and train model as two separate subsystems and couples them afterward with the wheel-rail interaction, like for example the time step iteration method (TSI).

Liu [14] described the SCM and LCM methods in his work and subsequently introduced a new method, named the loosely coupled non-iterative algorithm. In the new method, a small time step is considered to use the already calculated displacements from the wheelset to determine the forces $P_v$ that act on the vehicle. The new wheelset displacements are obtained by solving the motion equation of the vehicle with acting force $P_v$. The force $P_b$ acting on the bridge can be calculated with the considered wheel-rail interaction and the new wheelset displacements. Finally, the displacements of the bridge are determined by solving the motion equation of the bridge with the force $P_b$ acting on the bridge. If it is assumed that there is always contact between the wheels and track, then the calculated displacements of the bridge are equal to the new wheelset displacements and the iteration loop is repeated. In this method there is no convergence criteria to stop the iteration loop, only a predefined end time. So when the end time is reached, the iteration loop is stopped.

Liu [14] also compared the previous three iteration methods: SCM, LCM and loosely coupled non-iterative algorithm. For his numerical validation and comparison, he used a simply supported beam subjected to a train type ETR500Y. The displacement and accelerations of the bridge and train are approximately the same for each iteration method. In figure 2.13 the error of the vertical displacements is plotted for the different iteration methods in function of the number of time steps $N_t$ per period $T_b$. The LCM is compared for two acceptable tolerances (Tol) on the convergence criteria, namely $10^{-6}$ and $10^{-8}$. The latter one is more strict, so the results should be more accurate. The SCM and LCM (Tol = $10^{-8}$) become very accurate when increasing the number of time steps $N_t$ per period. The LCM for both tolerances reaches a minimum error for a specific $N_t$. When further increasing $N_t$, the accuracy of the vertical displacements decreases. The loosely coupled non-iterative algorithm has the lowest accuracy, even when increasing $N_t$. It can be concluded that even when the least accurate algorithm is used, only an error of 0.01% is made on the vertical displacements of the bridge. When interested in very local detail, it would be recommended to use the SCM or the LCM with strict tolerance. When only interested in the global deformation of the bridge, the loosely coupled non-iterative algorithm would be sufficient.
Zhang and Xia [53] introduced the inter-system iteration method (ISI) to solve the vehicle-bridge system shown in figure 2.14. In this method, the subsystem of the train and bridge are solved separately each time step over the whole time domain. This inter-system iteration method and the loosely coupled non-iterative algorithm differ only by the stop criteria for the iteration loop. The ISI method has a convergence criterion that says that the wheel-rail interaction forces in the current loop should be approximately the same as the wheel-rail interaction forces in the previous iteration loop within an acceptable tolerance. Zhang and Xia [53] compared the vertical, lateral and torsional displacement at midspan of a simply supported prestressed beam for the ISI method with the TSI method, namely a type of the LCM. They can conclude that the results for the ISI method are similar to those from the TSI method.

Zhu et al. [18] presented a combination of the LCM and SCM, named the hybrid solution algorithm (HSA), where the train and track are coupled as one subsystem and the bridge as a second subsystem. The SCM was used to solve the motion equations of the train-track subsystem and the LCM was used to couple both subsystems. They also applied the previous three models to a case study and compared the results. The solution algorithms are applied to the case study of a cable-stayed bridge located between Shanghai and Kunming which is adopted to an ICE-3 high speed train. The accuracy in displacements and accelerations of the bridge and vehicle obtained with the different models were
found to be very close to each other. When there is no significant difference between the results of the considered models, it becomes interesting to compare the differences in calculation times. Table 2.1 shows the results obtained in this case study. The hybrid model is 4 times faster than the LCM, but the hybrid model cannot reach the same computational efficiency when similar time steps as are used as in the LCM. The hybrid model is 2.5 times faster than the SCM, because in the hybrid model, only the elements that are time dependent are updated, whilst in the SCM all the elements are updated each time step.

<table>
<thead>
<tr>
<th>Method</th>
<th>Elapsed time [s]</th>
<th>Time step [s]</th>
<th>Total integration steps</th>
<th>Calculation time per step [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>LCM</td>
<td>1546</td>
<td>0.0002</td>
<td>40000</td>
<td>0.039</td>
</tr>
<tr>
<td>SCM</td>
<td>965</td>
<td>0.001</td>
<td>8000</td>
<td>0.121</td>
</tr>
<tr>
<td>HSA</td>
<td>379</td>
<td>0.001</td>
<td>8000</td>
<td>0.047</td>
</tr>
</tbody>
</table>

Table 2.1: Comparison of the different parameters and calculation time for the LCM, SCM and HSA

Neves et al. [54] established a solution algorithm for non-linear vertical wheel-rail interaction systems, called the optimized block factorization algorithm. This method is validated in his work by comparing the block factorization algorithm with the Lagrange multiplier method for the displacements, accelerations and contact forces of the bridge and vehicle model. It is a 2D model, so lateral interaction forces cannot be considered.

### 2.6 Thesis goal

The main objective of this thesis is to build upon previous research [55] [56] by expanding the analysis of a steel through-truss bridge. In the initial two master theses, a global finite element model of the bridge was created and validated using strain measurements obtained from the physical structure. However, the validation was limited to quasi-static load cases, neglecting dynamic load effects.

In the subsequent third dissertation [22], a novel approach was introduced, focusing on the development of a vehicle-structure interaction (VSI) model. Preliminary results indicate that the strain ranges obtained from the VSI model align more closely with experimental measurements compared to those obtained from quasi-static loads, which is also found in section 2.1. As summarized in the previous literature review, there is already a wide range of possible vehicle models available, each varying in complexity. However, it is important to note that as the complexity of the model increases, the computational cost reaches its limitations.

The main goal of this dissertation is to model and validate a train-bridge interaction model, which has a good balance between accuracy, complexity and computational cost.
Chapter 3

Developing a dynamic train-bridge interaction model

In the following chapter, a dynamic train-bridge interaction model is developed. From literature it is found that a spring-damper-mass model would be accurate enough compared to the more complex train models to determine the vertical deformations of the bridge at train speeds different from the critical speed. Two modelling approaches are compared: the direct integration method and the intersystem iteration method. First, both principles will be explained, and afterward, the methods are applied to different cases. The first case considers a simply supported three-dimensional beam loaded with a moving spring-mass model and will be validated with the analytical calculation. The second and third cases treat the passage of respectively the Swedish green train and the Italian high-speed train ETR500Y modelled as a series of spring-damper-mass models. The cases will be validated by comparing them to simulations found in the literature. In addition, it was noted that their complexity increases significantly.

Finally, both methods are compared whilst bearing in mind their useful application area.

3.1 Methodology

The goal of the dissertation is to develop a dynamic wheel-rail interaction model to predict the stresses in the bridge during the passage of a train. The interaction between the wheel and rail is solved by two different methods: the direct integration method and the intersystem iteration method. Both methods are compared with each other in accuracy and computational efficiency. Normally, both methods only differ in approaching the problem in a mathematical way, indicating the main difference would be in the computational efficiency. The physical interpretation of both methods should be the same, allowing only a slight difference in accuracy.

For the direct integration method, the train model and bridge model are solved together and are in direct interaction with each other. The iteration procedure is given in figure 3.1. In each time step, the bridge and train equation are iteratively solved until convergence. The convergence check means that the difference between the wheel-rail interaction forces of the $i^{th}$ iteration loop and the $(i+1)^{th}$ iteration loop is smaller than a predefined convergence check. In this dissertation, a convergence check for the wheel-rail interaction forces of 10 N is taken. Calculation of the next time step $t + \Delta t$ starts if the convergence check in the previous step is satisfied.
In the intersystem iteration method, the system is divided into two subsystems: the bridge and train subsystem. Both subsystems are coupled through the wheel-rail interaction forces but are separately solved as explained in section 2.5. The iteration procedure of the intersystem iteration method is given in figure 3.2. First, the train system is excited by some random track irregularities while the motion of the bridge is considered to be zero. The motion of the bogie and the wheel-rail interaction forces due to these irregularities are determined. These forces are then applied to the bridge model to determine the deflection of the bridge. Finally, it is assumed that the wheels and rails always have contact, so the motion of the bridge can be used as input motion to excite the train model again. This calculation loop is finalized by a convergence check, so the difference between the wheel-rail interaction forces calculated in step i and step i-1 must be smaller than the predefined convergence check of $10^N$. 

![Figure 3.1: Iteration loop of the direct integration method (DIM)](image)
The main objective of this research is to investigate the global behavior of the bridge structure during the passage of a train. It is important to acknowledge that the accuracy of the contact surfaces between the wheel and the bridge might be compromised due to the assumptions made during the analysis.

Assumptions:

1. The wheel and rail always have contact with each other
2. The train moves at a constant speed over the bridge, so it is assumed that the train does not accelerate on the bridge.
3. Only the vertical wheel-rail interaction is considered, the lateral and longitudinal interactions are neglected.
4. No track irregularities are taken into account, the rails are assumed to be perfect.
5. The components of the vehicle are considered rigid, so the elastic deformation during vibration is neglected
6. The track and the bridge are considered as one whole element So there is no interaction between the track and the bridge.
3.2 Validation of the modelling approach

As described in the previous section 3.1, two solving methodologies are simulated. The two modelling methods need to be validated to have knowledge about the further use of the models in real experiments. The first approach involves creating a simplified numerical 3D model in Abaqus for a simply supported three-dimensional beam. The beam is assumed to have a uniform stiffness $I$ and mass density $m$ along its length.

Both methodologies aim to determine the vertical deformation at the midspan of the beam. The results of both simulations are compared with each other to assess the accuracy. Therefore, the root mean square error (RMSE) and the relative root mean square error (RRMSE) are used. The RMSE gives the absolute root mean squared deviations, while the RRMSE given by equation 3.1 is the normalized RMSE and can therefore be used to compare different measurements. The predicted values are given by $\hat{y}_i$ and are compared with the real values $y_i$ for $n$ samples. Further, the calculation time is also an important parameter. Finding the optimal balance between duration and accuracy is a key focus for the comparison of both methods.

$$RRMSE = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{\sum_{i=1}^{n} \hat{y}_i^2}}$$

(3.1)

Based on the literature review (see section 2.4), the stresses in the beam are accurate enough to model the train as a series of independent spring-damper-mass models at non-critical speeds. Thus, in this dissertation, the train model is also simplified to independent spring-damper-mass models. This results in a decrease in the complexity and an increase in the calculation speed.

In section 3.3 a moving spring-mass model is applied to the simply supported beam. The numerical models are validated against an analytical solution obtained by solving the general differential equation of motion 2.2. Additionally, damping is incorporated into the models from section 3.4, and their validation is based on relevant research papers.
3.3 Case A: simply supported beam with moving spring-mass model

By first looking at a simple case the main defects of both methods can be detected. The first case that is considered is one spring-mass model moving along a simply supported beam as shown in figure 3.3. The properties of the beam and spring-mass are summarized in table 3.1 and are based on the first numerical example of Dinh-Van Nguyen et al. [57]. The beam has a length \( L \) of 30 m and a squared cross-section with a side length of 3 m. The elastic rigidity and density are respectively given by \( EI = 19.07 \cdot 10^{10} \text{ Nm}^2 \) and \( \rho = 5400 \text{ kg/m}^3 \). One end of the beam is hinged and the other end of the beam is supported by a roll. The beam is considered undamped and also the damping of the vehicle model is taken to zero. The stiffness of the bogie is \( K_V = 1060 \text{ kN/m} \) and the mass of the wheel is ignored. Neglecting the mass of the wheel and the damping simplified the analytical equation so that it is possible to solve it by hand without making approximations.

![Figure 3.3: Case A: simply supported beam loaded with a moving spring-mass model](image)

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>30</td>
<td>m</td>
</tr>
<tr>
<td>( A )</td>
<td>9</td>
<td>m²</td>
</tr>
<tr>
<td>( E )</td>
<td>28.25</td>
<td>GPa</td>
</tr>
<tr>
<td>( I )</td>
<td>6.75</td>
<td>m⁴</td>
</tr>
<tr>
<td>( \rho )</td>
<td>5400</td>
<td>kg/m³</td>
</tr>
<tr>
<td>( \bar{m} )</td>
<td>600</td>
<td>kg/m</td>
</tr>
<tr>
<td>( M_1 )</td>
<td>42550</td>
<td>kg</td>
</tr>
<tr>
<td>( K_V )</td>
<td>1060</td>
<td>kN/m</td>
</tr>
<tr>
<td>( h_1 )</td>
<td>1.5</td>
<td>m</td>
</tr>
<tr>
<td>( V(t) )</td>
<td>15</td>
<td>m/s</td>
</tr>
</tbody>
</table>

Table 3.1: Properties of the beam and spring-mass model

3.3.1 Analytical solution

The general differential equations of motions 2.2 can be rewritten to equations 3.2, 3.3, 3.4, 3.5. The first equation 3.2 describes the motion of the spring-mass model. The mass of the wheel \( M_1 \) is neglected in the first model and only the mass of the bogie \( M_2 \) is considered. The interaction between the wheel and the beam is described by the wheel-rail interaction forces \( P(t) \). The spring-mass model has a constant velocity \( V(t) \). The stiffness \( K_V \) of the spring transfers the loads between the beam and the bogie. The damping constant is neglected in this case to simplify the analytical equations.

The motion of the bridges is given by equation 3.3, where \( q(t) \) represents the modal coordinates of the beam. The geometrical coordinates of the bridge’s deformation are determined by the summation
of the modal coordinates multiplied by the respective mode shapes for each mode. Figure 3.4 illustrates the analytical results of the vertical deformation at midspan of the beam caused by a moving spring-mass model. These results were obtained considering the inclusion of one to four modes. It is noteworthy that the solutions for one mode and four modes are nearly identical. However, the root mean square error increases by including an extra mode and an error of $RMSE = 1.18 \cdot 10^{-5}$ is made if only one mode is considered instead of four. Due to this small error, the findings confirm that accurate results can be achieved by considering only the first mode of vibration.

$$M_2 \ddot{Z}(t) + K_V [Z(t) - y(x, t)|_{x=s(t)}] = 0$$  \hspace{1cm} (3.2)

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = \frac{2}{m L} P_i(t)$$  \hspace{1cm} (3.3)

$$P_i(t) = [M_2 g + K_V z(t) - q(t) K_V \sin(i\pi x/L)] \sin(i\pi x/L)$$  \hspace{1cm} (3.4)

$$y(x, t) = \sum_{i=1}^{n} q_i(t) \sin(i\pi x/L)$$  \hspace{1cm} (3.5)

Figure 3.4: The analytical results of the vertical displacements of a simply supported beam at midspan in time due to a moving spring-mass model, taking into account 1 to 4 modes.

The analytical model is based on the Euler-Bernoulli Beam Theory, while the numerical simulations perform a 3D model and consider the Timoshenko Beam Theory. The shear stresses in the Euler-Bernoulli Beam Theory are neglected, which results in a higher bending stiffness compared to the Timoshenko beam if the same beam is considered. Therefore, the bending stiffness of the Euler-Bernoulli beam in the analytical simulation must be adapted in order to have a valuable comparison. Kazimir [22] showed in section 3.1.3 of his master dissertation how to adapt the bending stiffness of the analytical calculation to have the same approach as the numerical model. The change in stiffness is based on the static vertical deflection at midspan given in equation 3.6 due to a vertical downward point load $P$ at midspan. A static general calculation in Abaqus of the beam is performed under a point load $P = 1000 \, N$. This results in a static vertical deformation at midspan of $y_{static} = -3.26179 \cdot 10^{11} \, m$ and an adapted stiffness of the beam $I = 6.1045 \, m^4$. In the following sections, the adapted stiffness is used for the analytical model.
\[ y_{\text{static}}(L/2) = \frac{PL^3}{48EI} \]  

(3.6)

### 3.3.2 Numerical solution: Abaqus

#### 1. Direct integration method

In the direct integration method, the beam and spring-mass model are modelled together. They are strongly coupled and are in constant interaction with each other during the calculation. The boundary conditions at the contact surface change during the calculation, depending on the motion of the beam and wheel.

The spring-mass is modelled as two points vertically above each other, referred to as the bogie and the wheel. The distance between the bogie and wheel is defined by \( h_1 \). The mass of the wheel is neglected, so only the bogie is given the inertia property with mass \( M_2 \). For the direct integration method, it is important to model contact between the wheel and the beam. The surface-to-surface contact interaction in Abaqus is used between the wheel and the top surface of the beam. The normal contact between both elements is modelled as hard contact, because the wheel and beam are hard materials and would not penetrate very deep into each other. The tangential contact is considered frictionless. No irregularities are taken into account and the elements are assumed to be perfect. To guarantee that the wheel and beam are always in contact, the DLOAD subroutine in Abaqus is used.

To activate the system, it is necessary to first have a static general step where the complete model is loaded only with gravity. The duration of the gravity step is taken one second. In this step, the motion of the bogie and the wheel are fixed in the horizontal direction. In the second step, a dynamic implicit calculation is performed and the boundary conditions of the bogie and wheel change. The duration of the second step depends on the time the spring-mass model needs to pass the beam. The bogie and wheel are not fixed anymore in the direction of movement along the beam. In the transverse direction of motion, the displacements of the bogie and wheel are still fixed, meaning only the vertical motion of the beam and vehicle are considered and any other lateral effects that could occur are neglected. The bogie has a constant velocity \( V \) along the beam.

The mesh of the beam has an influence on the accuracy and calculation time of the method. Therefore, the simulations are performed for different mesh sizes to compare the results. Figure 3.5 shows the vertical deformation at midspan in time for the considered model. The calculation time and accuracy for each mesh are compared in table 3.2. The accuracy is given by the root mean square error, where the error is calculated by the difference between the simulation of the considered mesh size and these with mesh size \( 0.1 \) m. A mesh of \( 0.1 \) m would have the most accurate results, but the increase in calculation time is very large and is therefore not preferred. Increasing the mesh size to \( 0.2 \) m or \( 0.3 \) m decreases the calculation time with a factor of 40 to 80. An uneven mesh size has a more accurate result than an even mesh size. This is because the beam has a width of \( 3 \) m and the moving spring-mass model is considered to move along the center of the beam. For an uneven mesh, the center coincides with exact nodes. While for an even mesh, the middle is in between two nodes, resulting in lower accuracy. It can be concluded that the optimum mesh size depends on the location where the wheels need to be acted on the beam. If possible, choose a mesh between \( 0.2 \) m and \( 0.3 \) m that divides the beam into nodes that coincide with the location of the wheel.
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Figure 3.5: Vertical deformation at midspan of a simply supported beam due to a moving spring-mass model for different mesh sizes

<table>
<thead>
<tr>
<th>Mesh size [m]</th>
<th>Calculation time [s]</th>
<th>RMSE [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>20</td>
<td>0.01163</td>
</tr>
<tr>
<td>0.5</td>
<td>45</td>
<td>0.001735</td>
</tr>
<tr>
<td>0.3</td>
<td>141</td>
<td>0.001015</td>
</tr>
<tr>
<td>0.2</td>
<td>312</td>
<td>0.001338</td>
</tr>
<tr>
<td>0.1</td>
<td>11738</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 3.2: Summary of the calculation time and accuracy of the simulations with different mesh size

II. Intersystem iteration method

The intersystem iteration method incorporates two subsystem models: a beam model and a spring-mass model.

The beam model represents a three-dimensional (3D) representation of a simply supported beam with a length $L$. The beam’s cross-section is square-shaped, and its material properties are listed in Table 3.1.

On the other hand, the spring-mass model comprises two interconnected nodes joined by a spring with a stiffness coefficient $k$. The nodes are positioned at a distance $h_1$ from each other. The upper node, referred to as the bogie, has an inertia characterized by a mass $M_1$, while the lower node represents the wheel. Initially, the spring-mass model is subjected to gravitational loading.

The intersystem iteration method iterates over different simulations performed with the software Abaqus. Therefore, a Python script is written to automate the calculation, which is added in appendix 4. The intersystem iteration method starts by exciting the spring-mass model with some random track irregularities to initiate the motion of the spring-mass model. These irregularities are given as initial vertical deformations of the wheel in time. The deformations of the bogie and wheel are fixed in the horizontal directions and only movement of the spring-mass model in the vertical direction is allowed.
Clauss and Schiehlen [58] generalized some random track irregularities into a power spectral density function (PSD), called the German Low Disturb Spectrum. This spectrum is based on measurements of the motion of the track due to the German High-speed train ICE and is given in equation 3.7

\[
S_V(\Omega) = \frac{A_v \Omega_r^2}{(\Omega^2 + \Omega_r^2)(\Omega^2 + \Omega_c^2)}
\]  

(3.7)

Where \(S_V(\Omega)\) gives the power density function of the vertical track irregularities in function of the frequency \(\Omega\). The frequency domain \(\Omega\) is based on the expected vertical motion of the beam, as exciting the spring-mass model with too large/small motions, this would result in more iteration loops before convergence is reached. The vertical roughness of the track is included by \(A_v = 1.58610^{-6} \text{ rad m}^{-1}\) and the parameters \(\Omega_r = 0.0206 \text{ rad m}^{-1}\) and \(\Omega_c = 0.8246 \text{ rad m}^{-1}\) are constant pre-defined factors. The power density function is given in figure 3.6a, with the inverse Fourier transformed to a time domain given by figure 3.6b.

![German Low Disturbance Spectrum - Vertical](a) Frequency domain  
![German Low Disturbance Spectrum - Vertical](b) Time domain

Figure 3.6: Plots of the PSD function for the vertical track irregularities

Then the vertical reaction forces in the wheel of the spring-mass model are exported. The loads are placed on the beam using the subroutine DLOAD in Abaqus. This subroutine allows applying of the reaction forces on the beam at specified locations defined by time. The DLOAD subroutine is added in appendix 4 and explains how the loads are specified on the beam. It must be taken into account that the DLOAD subroutine applies surface loads. An exact point load is not possible with DLOAD, so the loads are modified, depending on the mesh of the beam. The point load is distributed over the surface on which it acts. The smallest possible surface is a square with the side length of the mesh. So the smaller the mesh of the beam, the more the DLOAD approaches an exact point. To find the optimal mesh, beams with different meshes are subjected to a moving constant force of 500 kN. Figure 3.7 shows the vertical deformation at midspan for different mesh sizes. Meshes larger than or equal to 1 m give an unfavorable overestimation of the results and cannot be used. The calculation time for the different simulations is given in table 3.3. The simulation with a mesh size of 0.1 m is taken as a reference to calculate the root mean square error (RMSE) between the different simulations. A mesh of 0.1 m gives small discrepancies compared with the other calculations. These small effects are ignored by increasing the mesh size. Although the high accuracy of the calculation with a mesh of 0.1 m, the increase in calculation time is too large and not preferred. The optimum mesh for the simulation with the intersystem iteration method lies in the range of 0.2 m to 0.3 m. In general, the smaller the mesh, the higher the accuracy of the results. The position where the wheels act has no influence on the mesh size for the intersystem iteration method.
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Figure 3.7: Vertical deformation at midspan due to a moving point load for different meshes of the beam

<table>
<thead>
<tr>
<th>Mesh size [m]</th>
<th>Calculation time [s]</th>
<th>RMSE [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>21</td>
<td>0.01671</td>
</tr>
<tr>
<td>0.5</td>
<td>50</td>
<td>0.006115</td>
</tr>
<tr>
<td>0.3</td>
<td>132</td>
<td>0.004214</td>
</tr>
<tr>
<td>0.2</td>
<td>555</td>
<td>0.003833</td>
</tr>
<tr>
<td>0.1</td>
<td>113085</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 3.3: Summary of the calculation time and accuracy of the simulations with different mesh sizes

Finally, the vertical deformations at the bottom of the beam at the locations of the wheel are exported and again given as boundary conditions to the wheels. So the calculation loop restarts with the new vertical deformations of the beam instead of the track irregularities. The iteration loop goes further until convergence is reached. The iterations are limited by a convergence check of $10^{-N}$ on the wheel-rail interaction forces. Figure 3.8 proves the convergence of the intersystem iteration method. This graph shows the deformation at node 1498 at midspan for each iteration loop. The deformation at node 1498 is measured at the moment the wheel is at midspan, so after 1 s. In this case, convergence is reached after four iteration loops.
For this particular node and time step, the deformation at midspan in the first loop is already close to the end deformation. Figure 3.9 shows the vertical deformation at midspan in time due to a moving spring-mass model for the different iteration loops until convergence. This graph shows that the first loop already has a good deformation for each time step, so it is already close to the converged deformation. The root mean square error for the different loops compared with the converged loop four are given in Table 3.4. The errors between the different iteration loops are very small which led to the conclusion that the first iteration loop approaches the end results. If the calculation times increase significantly, it would be possible to calculate only the first or second iteration loop to have an idea of the behavior of the beam.
### 3.3.3 Comparison of the three methods

The intersystem iteration method (ISIM) and the direct integration method (DIM) are validated with the analytical solution. To compare both methods, it is necessary that the calculation accuracy is the same. From previous sections, the optimum mesh for the direct integration method is $0.3 \text{ m}$ and for the intersystem iteration method is $0.2 \text{ m}$. Therefore, both methods are simulated with their mesh according to the highest accuracy. The total time is $2 \text{ s}$, the time the spring-mass model needs to move over the beam at a speed of $15 \text{ m/s}$. A time increment of $0.02 \text{ s}$ is considered in both methods, to have exactly 100 increments. The vertical deformation at midspan of the beam in time is given in figure 3.10.

![Figure 3.10: Displacements at midspan for the three different calculation methods: Analytical, ISIM, DIM](image)

Initially, the results of the two methods seem close to each other compared to the analytical solution. The curves of the numerical solutions are shifted upward and have a lower maximal vertical deflection at midspan than the analytical results. As mentioned in section 3.3.1, the analytical calculations are based on the Euler-Bernouilli beam theory, while the numerical results are based on the Timoshenko beam theory. The stiffness in the analytical simulation is adapted to a lower stiffness based on the static deflection of the numerical beam model due to a point load. This way, it is allowed to compare the numerical results with the analytical ones but it is still an approximation. The relative root mean square error (RRMSE) between the simulations and the analytical results are given in table 3.5. Both methods differ $6 - 7\%$ of the analytical calculation. This is mainly caused by the difference in beam theory. The results of the ISIM are closer to the analytical simulation, but the total calculation time is 22 times larger.
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<table>
<thead>
<tr>
<th>Method</th>
<th>Calculation time [s]</th>
<th>RRMSE [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIM</td>
<td>141</td>
<td>0.07234</td>
</tr>
<tr>
<td>ISIM</td>
<td>3180</td>
<td>0.06092</td>
</tr>
</tbody>
</table>

Table 3.5: Comparison of the two methods in calculation time and accuracy

The difference between the Euler-Bernouilli beam theory and Thimoshenko beam theory becomes smaller by increasing the length of the beam. Abdarrhim and Abdussalam [16] discussed in their paper the difference between the Euler-Bernoulli beam theory and the Thimoshenko beam theory. Figure 3.11 shows the vertical deformation of a simply supported beam due to uniform distributed force along the beam. The deflections for a Thimoshenko beam are given in function of $t/L$, with $t$ the thickness of the beam and $L$ the length of the beam. The paper shows that the total deflection and rotation of the Thimoshenko beam depends on the ratio $t/L$, while the bending rotation and stresses are not affected by the length or thickness of the beam. This means that for larger beams the influence of the shear stresses becomes negligible small on the total bending rotation. Thus, by increasing the length of the beam $L$, the Thimoshenko beam approaches the Euler-Bernouilli beam.

Therefore, the different methods are also compared for a beam with length $L = 100 \text{ m}$. The ratio $t/L$ becomes 0.03 which should be sufficiently small to make the difference in approximation theoretically negligible. The same properties as given in table 3.1 for the beam with a length of 30 m are considered, only the length of the beam is adapted. Therefore, the stiffness of the beam in the analytical model will be changed and is determined again with a static general calculation under a point load $P = 10000 \text{ N}$ at midspan. The resulting static deflection at midspan is $y_{stat} = -1.10575 \cdot 10^3 \text{ m}$ and the adapted flexural rigidity of the beam is $I = 6.67 \text{ m}^4$. The vertical deformation at midspan for this beam, due to a moving spring-mass model, is given in figure 3.12. Again, the relative root mean square error between the two numerical methods and the analytical solutions are determined and given in table 3.6. The difference in accuracy between the two numerical methods is in the same order of 1% and the ISIM approaches the analytical simulation better. The error between the analytical and numerical simulations decreases remarkably by increasing the length of the beam. It can be concluded that the larger error for the shorter beam is due to the different approximations of Euler-Bernoulli and Thimoshenko. Also, the calculation time is calculated for both methods. The total duration is increased
compared with the shorter beam. This is because the same mesh for both beams is considered, so 3.33 more elements must be calculated.

![Displacements at midspan of a beam with length 100 m for the three different calculation methods: Analytical, ISIM, DIM](image)

**Figure 3.12:** Displacements at midspan of a beam with length 100 m for the three different calculation methods: Analytical, ISIM, DIM

<table>
<thead>
<tr>
<th>Method</th>
<th>Calculation time [s]</th>
<th>RRMSE [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIM</td>
<td>349</td>
<td>0.03208</td>
</tr>
<tr>
<td>ISIM</td>
<td>2160</td>
<td>0.01956</td>
</tr>
</tbody>
</table>

**Table 3.6:** Comparison of the two methods in calculation time and accuracy for the beam with length 100 m
3.4 Simply supported beam with a series of moving spring-damper-mass models

In previous models, the damping of the vehicle is neglected, but it cannot be further neglected once the behavior of a real train is considered. Therefore a model that includes damping must also be validated.

Arvidsson et al. [10] presented in their work the behavior of a simply supported beam subjected to the Swedish Green train. In the paper, they modelled the train as a series of moving spring-damper-mass models. To validate the intersystem iteration method and the direct integration method of this dissertation, the same model as in the paper is created and compared.

The paper takes a simply supported beam with a length \( L = 36 \text{ m} \) and a moment of inertia \( I = 0.820 \text{ m}^4 \). The exact cross-section of the beam in the paper is not mentioned, so in this dissertation, a beam with a squared cross-section is considered. The side lengths of the cross-section of the beam are taken that they approach really good the properties of the beam treated in the paper. Not only the damper in the vehicle creates damping, but also the bridge damps the motion of the train-bridge system. Rayleigh damping is considered, so the Rayleigh parameters \( \alpha_R \) and \( \beta_R \) must be determined for the discussed beam. The parameter \( \alpha_R \) is proportional to the mass of the beam and damps the lower frequencies, while the parameter \( \beta_R \) is proportional to the stiffness of the beam and damps the higher frequencies. The damping ratio \( \xi \) is 0.005 and is given by formula 3.8

\[
\xi_i = \frac{\alpha_R}{2f_i} + \frac{\beta_R f_i}{2} \tag{3.8}
\]

With \( f_i \) the natural bending frequency of mode \( i \). The first and second bending frequency of the considered beam is determined by a frequency analysis in Abaqus. The resulting deformation for the first and second bending frequencies are given in respectively figure 3.13 and 3.14.

![Figure 3.13: The first bending eigenfrequency of the beam, \( f_1 = 3.8588 \text{ Hz} \)](image)
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Figure 3.14: The second bending eigenfrequency of the beam, $f_2 = 14.756 \text{ Hz}$

From equation 3.8, the Rayleigh parameters $\alpha_R = 0.03059$ and $\beta_R = 0.0005372$ are determined. The resulting Rayleigh curve is given in Figure 3.15. The damping ratio is constant and crosses the Rayleigh curve in the first and second bending frequencies.

Figure 3.15: Rayleigh curve for the beam with modal damping ratio $\xi = 0.005$, first bending frequency $f_1$ and second bending frequency $f_2$

Table 3.7 gives the properties of the beam model in this work and in the paper. The paper assumes the Euler-Bernouilli theory, so the stiffness in the 3D model is increased with the same concept as in section 3.3.1 to approach an Euler-Bernouilli beam.
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<table>
<thead>
<tr>
<th>Property</th>
<th>Paper</th>
<th>Beam model</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>36</td>
<td>36</td>
<td>m</td>
</tr>
<tr>
<td>$E$</td>
<td>210</td>
<td>210</td>
<td>GPa</td>
</tr>
<tr>
<td>$I$</td>
<td>0.820</td>
<td>0.875</td>
<td>$m^4$</td>
</tr>
<tr>
<td>$m$</td>
<td>17000</td>
<td>17000</td>
<td>kg/m</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.5</td>
<td>0.5</td>
<td>%</td>
</tr>
<tr>
<td>$f_1$</td>
<td>3.86</td>
<td>3.8588</td>
<td>Hz</td>
</tr>
</tbody>
</table>

Table 3.7: Properties of the considered beam compared with the properties taken in the paper

The Swedish Green train load pattern is shown in figure 3.16 and further in this dissertation referred to as load pattern 1. The paper considered eight successive cars modelled as single spring-damper-mass models. The properties of the vehicles are given in table 3.8. The model takes the mass of the wheel $M_w$ and bogie $M_1$ into account. In the paper, they compare the results of a moving constant force, a moving spring-damper-mass model and a two-axle moving spring-damper-mass model. Therefore, the weight of the car $M_2$ is divided over the four wheels. This results in an additional axle load on the bogies of $F_{0.25\text{ car}} = 137.34\, kN$.

![Figure 3.16: Load pattern 1: The Swedish green train, dimensions in m](image)

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_2$</td>
<td>56000</td>
<td>kg</td>
</tr>
<tr>
<td>$M_1$</td>
<td>5000</td>
<td>kg</td>
</tr>
<tr>
<td>$M_w$</td>
<td>2000</td>
<td>kg</td>
</tr>
<tr>
<td>$K_V$</td>
<td>2000</td>
<td>kN/m</td>
</tr>
<tr>
<td>$K_{VV}$</td>
<td>600</td>
<td>kN/m</td>
</tr>
<tr>
<td>$C_V$</td>
<td>30</td>
<td>kN s/m</td>
</tr>
<tr>
<td>$F_{\text{car}}$</td>
<td>549.36</td>
<td>kN</td>
</tr>
<tr>
<td>$F_{0.25\text{ car}}$</td>
<td>137.34</td>
<td>kN</td>
</tr>
</tbody>
</table>

Table 3.8: Properties of the Swedish train

The given spring stiffness $K_V$ of the primary suspension system cannot be used for the spring stiffness of the spring-damper mass model. To compare the results between a system with one suspension and a system with two suspensions, the stiffness must be adapted. The bogie frequency of both vehicles should be the same and is given by equation 3.9. The bogie frequency of the two-axle vehicle is $4.8\, Hz$, this results in an equivalent spring stiffness $K_{V,\text{eq}} = 2273.956\, kN/m$ for the spring-damper mass model. Considering this stiffness of the spring-damper-mass model, the bogie frequency is determined with Abaqus and given in figure 3.17. The bogie frequency of the two-axle vehicle and the spring-damper mass model is exactly the same as the equivalent bogie frequency.
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\[ f_{vb} = \frac{1}{2\pi} \sqrt{\frac{2K_v + K_{vv}}{M_1}} \]  \hfill (3.9)

Figure 3.17: Abaqus result from a frequency analysis on the bogie, \( f_{vb} = 4.80 \text{ Hz} \)

In the paper, the train moves at resonance speed to have extreme excitation of the motion of the beam. The resonance speed \( V_{res} \) depends on the first bending frequency of the beam \( f_1 \) and the length of one repeated unit, the carriage length \( L_c \) of 26.6 m. The resonance speed for the considered beam model is 102.6 m/s. This resonance speed is much higher than the mean speed of regional trains in Belgium of 16 m/s and must be avoided in real situations. If the train speed is close to the resonance speed of the bridge, the train speed should be adapted when approaching the bridge to a lower speed to avoid resonance problems.

\[ V_{res} = f_1 L_c \]  \hfill (3.10)

3.4.1 Numerical models

In this section, the main problems that occur during the modelling of a train load pattern are described.

1. Direct integration method

The method described in section 3.3 is completely the same. The boundary conditions of the beam and spring model do not change by extending the length of the train. Also, the way of modelling contact between the wheels and beam is the same, hard frictionless contact. The biggest challenge in this method was approaching the bridge by vehicle models. It is chosen to model all individual spring-damper-mass models at the beginning above the support of the bridge. By employing this approach, potential issues in Abaqus regarding vehicle models falling down due to gravity before reaching the bridge are effectively mitigated. Also, the stiffness of the beam is not increased by extending the beam. Each time step \( \Delta t_i \) in the model refers to the start of the next spring-damper-mass model. The length of the time steps \( \Delta t_i \) is based on the distance between the wheels \( L_{inter} \) and the speed of the train \( V \).

\[ \Delta t_i = \frac{L_{inter}}{V} \]  \hfill (3.11)
During the first time step $\Delta t_1 = 1 \, s$, every spring-damper-mass model is fixed in the horizontal direction ($U_1 = U_3 = 0$) and only the bogie can move free in the vertical direction $U_2$. After loading the system with gravity, the spring-damper-mass models can each, in turn, start to move at time step $\Delta t_{1+i}$. The boundary conditions of the considered vehicle $i$ change in the corresponding time step $1 + i$. The motion in the longitudinal direction $U_3$ is no longer fixed and the vehicle can move along the beam. The rotations of the bogie and the wheel are in each time step fixed. Only the vertical interaction between the vehicle and the beam is considered. The lateral effects and irregularities are neglected. The vertical motion of the wheel equals the vertical motion of the beam. Figure 3.18 shows the orientation of the beam and the change in boundary conditions when the bogie $i$ starts to move in time step $\Delta t_{1+i}$.

![Figure 3.18: Visualisation of the boundary conditions for the spring-damper-mass models in the different steps](image)

**II. Intersystem iteration method**

To expand one moving spring-mass model to a series of spring-damper-mass models, extra vehicles should be modelled in the vehicle subsystem. The iteration loop remains the same. First, the 32 wheels will be excited by the German Low disturb spectrum. The time step where the excitation of each wheel starts is calculated in the same way as in the direct integration method (equation 3.11). A simplification should be made, because the ISIM is based on framesteps instead of time steps. Therefore, the time step $\Delta t_i$ is divided by the increment size and the wheels start to move after $X_i$ increments. This is an approximation of the real distance between the vehicles because a framestep needs to be an integer. A rounding of the real increment should be made. The error due to the rounding depends on the size of the time increment and the velocity of the train. The maximum error due to the rounding is one-half of the time increment and is summarized in table 3.9 for different sizes of the time increment. The size of the time increment also involves the total calculation time. A time increment of $0.005 \, s$ or smaller is preferable because the error becomes acceptably small. The disadvantage is the extreme increase in calculation time. By decreasing the time increment from $0.01 \, s$ to $0.005 \, s$, the calculation time is about two times larger. Therefore, a time increment of $0.005 \, s$ is not preferred anymore. Figure 3.19 gives the vertical deformation of the beam at midspan under load pattern 1 for the different sizes of time increment. The root mean square error between the different results is also given in the table, the smallest time increment of $0.005 \, s$ is considered as reference. The maximum possible error of $0.5139 \, m$ is large compared with the smallest inter-distance between wheels of $2.7 \, m$. Despite the large difference in maximum possible error, the resulting vertical deformations for a time increment of $0.01 \, s$ and $0.005 \, s$ are very close and only an error of $0.019\%$ is made. The considered time increment in further calculations is $0.01 \, s$. For larger time increments, the errors are not acceptable anymore.

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#### Time increment | Maximum error [m] | Calculation time [s] | Number of increments [-] | RMSE [%]
--- | --- | --- | --- | ---
0.1 | 5.139 | 2820 | 30 | 0.2940
0.05 | 2.5695 | 6480 | 60 | 0.3268
0.01 | 0.5139 | 44160 | 300 | 0.01905
0.005 | 0.25695 | 80760 | 600 | 0.0

Table 3.9: Summary of the maximum possible error in \( m \), the calculation time and number of increments in function of the size of the time increment.

![Graph showing vertical deformation of the beam at midspan under loading pattern one of MSDM for different time increment sizes](image)

Figure 3.19: Vertical deformation of the beam at midspan under loading pattern one of MSDM for different time increment sizes.

#### 3.4.2 Discussion of the results

Finally, to validate the simulations, the results of both methods are compared with those from the paper [10]. The vertical deformation and vertical accelerations of the beam at midspan are defined in the paper. In both cases, the horizontal axis is normalized by the time the last wheel leaves the beam \( t = 2.37 \text{ s} \).

The vertical deformation at midspan of the beam under loading pattern 1 of the moving spring-damper-masses is given in graph 3.20. The amplitude of the vertical deformations builds up with the passage of the vehicle. At the moment the train leaves the beam, an upward shift of the vertical deformations occurs. The loading on the beam releases and the beam keeps vibrating without any loading. The
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free response of the beam goes further until the beam returns back into static equilibrium. The calculation time and accuracy compared with the paper are given in table 3.10. As expected from the previous section, the calculation time of the ISIM is much longer than the DIM. Despite an increase in complexity, the DIM is notably faster in calculation time. On the other side, the ISIM shows significantly better accuracy.

Figure 3.20: Vertical displacements at midspan of the beam under a series of moving spring-damper-mass models (MSDM) under load pattern one with velocity $V = 102.78$ m/s, for the DIM, ISIM and the paper [10].

Figure 3.21 shows the vertical accelerations occurring at midspan in the beam under the given loading pattern 1 of the vehicles. The results show the same discrepancies as illustrated by the vertical deformations. The calculation time and root mean square error for these calculations are given in table 3.10. The calculation time remains the same because the acceleration and deformations are calculated at the same moment in Abaqus. The difference in accuracy between both methods is equivalent, but the error with the reported results from the paper becomes larger. The reason for this difference can be attributed to the larger irregularities in the vertical accelerations observed in the paper, which are not apparent in the simulation results.
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Figure 3.21: Vertical acceleration of the beam at midspan under a series of moving spring-damper-mass models (MSDM) under load pattern one with velocity $V = 102.78 \, m/s$, for the DIM, ISIM and the paper [10]

<table>
<thead>
<tr>
<th>Method</th>
<th>Calculation time [s]</th>
<th>RRMSE [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Displacement</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DIM</td>
<td>740</td>
<td>0.3834</td>
</tr>
<tr>
<td>ISIM</td>
<td>44160</td>
<td>0.2274</td>
</tr>
<tr>
<td><strong>Acceleration</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DIM</td>
<td>740</td>
<td>0.4491</td>
</tr>
<tr>
<td>ISIM</td>
<td>44160</td>
<td>0.2769</td>
</tr>
</tbody>
</table>

Table 3.10: Comparison of the two methods in calculation time and accuracy for the moving constant force model and the spring-damper mass model

The beam is loaded with a train that moves at resonance speed. This means that the beam will vibrate at its first eigenfrequency. Graph 3.22 shows the amplitudes in function of the frequency for the vertical deformation of the beam at midspan. The exact frequency of the vertical deformations is $3.8 \, Hz$. This is only a slight difference of $0.058 \, Hz$ with the eigenfrequency of the beam. The second, third and fourth eigenfrequencies of the beam are also visible on the orange curve of the direct integration method (DIM). The magnitude of the peaks of both methods differs significantly from each other. The amplitudes at the first eigenfrequency of the vertical motion of the beam for the DIM and ISIM are respectively $0.00460 \, mm$ and $0.00316 \, mm$. This difference is also remarkable on the graph of the vertical deformation and accelerations.
Developing a dynamic train-bridge interaction model

Figure 3.22: Frequency response of the results of the vertical deformations of the beam for the direct integration method (DIM) and the intersystem iteration method (ISIM)

The larger vertical amplitudes due to the DIM compared with the ISIM can have different reasons. In the ISIM, the loads that act on the beam are distributed over a small surface depending on the mesh size. The larger the mesh size, the more the loads are distributed over the beam. This results in less high point loads that are more distributed over the beam. However, it should be noted that the vertical deformations are extracted from the bottom of the 3D beam. In general, the forces will be distributed within the beam, resulting in a negligible influence.

The difference in amplitude depends also on the magnitude of the wheel-rail interaction forces. Therefore, the interaction forces between the spring-damper and the beam are determined for four randomly chosen wheels (Wheel 1, 10, 20 and 30), given in figure 3.23. For the intersystem iteration method (ISIM), the inverse of the vertical reaction forces (RF2) in the wheel of the vehicle subsystem is given. In the direct integration method, the total vertical contact forces (CFT2) between the beam and the wheel are given. The magnitude of the wheel-rail interaction force for both methods is in the same order. The initial wheel-rail interaction force at time step \( t = 0.0 \, s \) is approximately 45 kN and refers to the sum of the inertia of the wheel and bogie. Immediately, the wheel starts to move, the axle load is subjected to the bogie. The wheel-rail interaction forces of all the wheels build up until the beam passes one-third of the beam. Afterwards, the wheel-rail interaction forces bounce around the equilibrium of 180 kN, the sum of all the masses of the vehicle. For wheel 10 and 30 a horizontal shift of the wheel-rail interaction forces is visible in the DIM compared with the ISIM. The wheel-rail interaction force in the intersystem iteration method is not directly affected by the other vehicle models. Unlike the DIM where the different vehicles influence each other. For the first wheel, the reaction forces of both methods are very similar because the first wheel cannot be affected by a series of previous wheels. The position of the wheels in the train is given in figure 3.16. This shows that at the connections of a new train car, four wheels pass fast after each other. Between the rapid succession of four wheels, there is a longer period of time. This means that the first wheel in the sequence of four will experience less influence from the previous wheels than the last wheel in the sequence of four. Wheel 10 and wheel 30 are both the fourth wheel in the sequence of four, so already 3 wheels are passed close after each other. Wheel 20 is just the second wheel in the sequence of four wheels. This influence of the previous wheels that only occurs in the direct integration method (DIM) declares the difference in wheel-rail interaction forces for the different wheels that are displayed.
Developing a dynamic train-bridge interaction model

Figure 3.23: Comparing the vertical wheel-rail interaction forces for both methods in the wheels 1, 10, 20, 30.

As explained, the beam and vehicle model are directly coupled with each other in the DIM. This means that the motion of the first wheel has a direct influence on the motion of the second wheel. So, a small error could give a cascading effect and should explain the increasing error between the DIM and the ISIM during the passage of the train. This effect was already visible in the wheel-rail interaction forces. Therefore, the system is simplified again and both methods are compared for one moving spring-damper-mass system. The properties of the vehicle are the same as considered for the whole train and are given in Table ???. The vertical deformation at midspan of the simply supported bridge under one moving spring-damper-mass model is given in Figure 3.24a. Also two and three moving spring-damper-mass systems are considered to determine the influence by increasing one vehicle on the vertical deflection of the beam at midspan. The vertical deformation at midspan for two vehicles is given in Figure 3.24b. In the first case, where only one wheel is considered, the deformations given by the ISIM are slightly larger than those by the DIM. Considering the case of two wheels, the amplitudes of the DIM are becoming slightly larger than the ISIM. When once again increasing the number of wheels, the amplitudes of the DIM increase further and differs more from the ISIM. Table 3.11 summarizes the size of the amplitude of the vertical deformation for considering a different number of wheels. This shows that the amplitude simulated by the DIM becomes larger compared to the ISIM by increasing the number of wheels. The influence of the different wheels on each other becomes very large and is not negligible anymore. The increase in amplitude of the DIM would be due to the huge number of wheels. By increasing the number of wheels, the influence of the different wheels on each other becomes significant, resulting in an increase in amplitude.
Developing a dynamic train-bridge interaction model

Figure 3.24: Vertical deformation at midspan of the simply supported beam in function of time considering a different number of wheels passing the beam, \( V = 102.78 \, m/s \)
Table 3.11: Summary of the amplitude of the vertical deflections of the beam at midspan for a different number of wheels, found by the DIM and ISIM

<table>
<thead>
<tr>
<th>Number of wheels</th>
<th>Amplitude DIM [m]</th>
<th>Amplitude ISIM [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000706</td>
<td>0.000762</td>
</tr>
<tr>
<td>2</td>
<td>0.00133</td>
<td>0.00120</td>
</tr>
<tr>
<td>3</td>
<td>0.00174</td>
<td>0.00120</td>
</tr>
<tr>
<td>32</td>
<td>0.00460</td>
<td>0.00316</td>
</tr>
</tbody>
</table>

Due to the small influence of the previous wheels at the start in the DIM, the results of both simulations are similar. Both simulations overestimate the deformations and accelerations of the beam at the beginning compared with the paper. In the models considered in this dissertation, the wheels start to move at the first support of the bridge. This means that the wheels do not approach the beam, but start to move on the beam itself. The approaching of the beam by the vehicles is not explained in the paper. If the vehicles approach the beam instead of starting at the support, they are more stable at the moment they reach the beam, resulting in smaller deformations at the start.
3.5 Simply supported beam with a series of two-axle vehicles

In this section, the influence of approaching a train as a spring-damper-mass model is verified. In the handbook 'Bridge vibration and controls' of Xia et al. [17], different ways of modelling train-bridge interactions models are discussed and the influence of different model parameters on the results is analyzed. They treat three different vehicle models as shown in figure 3.25: model (a) has one degree of freedom (DOF) and equals the spring-damper-mass model discussed in this dissertation, model (b) has two degrees of freedom and is referred to the two-axle vehicle model as in the dissertation of Kazimir [22] and model (c) represents a full train car having three DOFs.

Figure 3.25: Visualisation of the three different vehicle models: (a) moving spring-damper-mass model (MSDMM), (b) two-axle vehicle bridge model (TAVBM), (c) Train-bridge dynamic interaction model (TBDIM) [17]

The handbook gives the results for model (c) for a simply supported beam with length \( L = 34 \, \text{m} \). Previous year, Kazimir simulated the results of model (b) for the same beam. To validate if a moving-spring-damper-mass model is accurate enough to determine the vertical deformations in a bridge while passing a train, the model is compared for both methods with the results from the handbook and Kazimir. Therefore, the beam and vehicle model as mentioned in the handbook are acquired. The beam has a length \( L = 34m \), flexural rigidity \( EI = 9.92 \cdot 10^{10} \, \text{N/m}^2 \) and density mass per unit length \( \rho = 11400 \, \text{kg/m} \). Transverse effects are again neglected, so the 3D model of the beam in Abaqus will be adapted, as in previous section to approach the Euler-Bernoulli beam. Figure 3.26 and 3.27 give the first and second bending frequencies of the beam. The damping parameters \( \alpha_R = 0.124 \) and \( \beta_R = 0.00226 \) are calculated as in previous section based on the Rayleigh model with a modal damping ratio of 0.02. The properties of the beam mentioned in the handbook are compared with these of the 3D model in table 3.12.
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Figure 3.26: First bending frequency of the considered beam, $f_1 = 4.0097$ Hz

Figure 3.27: Second bending frequency of the considered beam, $f_2 = 13.683$ Hz
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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Handbook</th>
<th>3D Beam model</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>34</td>
<td>34</td>
<td>m</td>
</tr>
<tr>
<td>$E$</td>
<td>$3 \cdot 10^{10}$</td>
<td>$3 \cdot 10^{10}$</td>
<td>GPa</td>
</tr>
<tr>
<td>$I$</td>
<td>3.307</td>
<td>3.658</td>
<td>m$^4$</td>
</tr>
<tr>
<td>$A$</td>
<td>4.75</td>
<td>6.625</td>
<td>m$^2$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>2400</td>
<td>1720.631</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.02</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>$f_1$</td>
<td>4.01</td>
<td>4.0097</td>
<td>Hz</td>
</tr>
</tbody>
</table>

Table 3.12: Properties of the beam mentioned in the paper compared with the 3D model in this dissertation

The train configuration of the Italian high-speed train ETR500Y considered in the paper is given in figure 3.28, which specified the number of carriages and their corresponding total axle load. The total train has eight passenger cars with an extra locomotive at the front and rear. The locomotive cars are heavier and shorter than the passenger cars. A 2D sketch of one passenger car is given in figure 3.29 with the properties for the locomotive and passenger car summarized in table 3.13. A 2D model of the train is considered, so the spring stiffnesses and the damping coefficients are multiplied by two. In this way, the same total stiffness and damping of the 3D model are approached. The considered spring-damper-mass model has a bogie mass $M_1/2$, a wheel mass $M_w$ and an axle load $F_{0.25 \text{ axle}} = g \cdot M_2/4$. The damping coefficient remains the same as in the complete train-bridge dynamic interaction model (c).

Figure 3.28: Load pattern 2: Train configuration of the Italian high-speed train ETR500Y

Figure 3.29: Sketch of the passenger car of the Italian high-speed train ETR500Y
Table 3.13: Properties of the Italian high-speed train ETR500Y

<table>
<thead>
<tr>
<th>Property</th>
<th>Locomotive</th>
<th>Passenger car</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_2$</td>
<td>55976</td>
<td>34231</td>
<td>kg</td>
</tr>
<tr>
<td>$M_1$</td>
<td>3896</td>
<td>2760</td>
<td>kg</td>
</tr>
<tr>
<td>$M_w$</td>
<td>2059</td>
<td>1583</td>
<td>kg</td>
</tr>
<tr>
<td>$K_V$</td>
<td>896100</td>
<td>404370</td>
<td>N/m</td>
</tr>
<tr>
<td>$K_{VV}$</td>
<td>236030</td>
<td>90277</td>
<td>N/m</td>
</tr>
<tr>
<td>$C_V$</td>
<td>7625</td>
<td>3750</td>
<td>Ns/m</td>
</tr>
<tr>
<td>$C_{VV}$</td>
<td>18125</td>
<td>8125</td>
<td>Ns/m</td>
</tr>
<tr>
<td>$h_1$</td>
<td>0.915</td>
<td>0.7</td>
<td>m</td>
</tr>
<tr>
<td>$h_2$</td>
<td>0.098</td>
<td>0.12</td>
<td>m</td>
</tr>
<tr>
<td>$h_3$</td>
<td>0.087</td>
<td>0.13</td>
<td>m</td>
</tr>
</tbody>
</table>

To compare the results of the handbook with the simulations in this dissertation with the moving spring-damper-mass model, the bogie frequency of both vehicle models must be the same. Therefore, the vertical stiffness of the primary suspension system is adapted with the formula 3.9 as mentioned in section 3.4. The properties of the locomotive car and the passenger car are different, so the bogie frequency would not be the same. The equivalent stiffness for the locomotive car and passenger car are respectively $K_{V_{eq,loc}} = 2028230.0 \, N/m$ and $K_{V_{eq,pas}} = 899017.0 \, N/m$. The resulting frequencies of the spring-damper-mass models are given in figure 3.30, which is exactly the same value as given in the handbook.

3.5.1 Numerical models

The construction of the numerical models remains the same as in section 3.4.1. Only the size of the time steps changes in both methods depending on the considered train speed and distance between the successive wheels. Additionally, eight extra wheels are modelled to have a train of 40 wheels in total.

3.5.2 Discussion of the results

The vertical deformation of the beam due to a moving train with load pattern 2 is given in figure 3.31. The DIM and ISIM assume the train as a series of moving spring-damper-mass models, while the paper models the different carriages as a whole like the TBDIM shown in figure 3.29. The calculation time
and accuracy compared to the paper are given in table 3.14. The calculation time of the DIM is again a lot faster but overestimates the deformations incredibly hard. The relative root mean square error made by the direct integration method at resonance speed for the vertical deformations at midspan is 61.28 %, which is not acceptable anymore. The error made by the ISIM is half as large as the error of the DIM, but is significantly increased compared to previous model. The same discrepancies of the simulations as in section 3.4.2 are visible. The amplitudes of the DIM are much higher compared to the ISIM, but the frequency is still in the same order. As was described, both simulations are closer to each other in the beginning due to the small influence of other wheels in the direct integration method. In the beginning, the ISIM gives larger deformations compared to the paper. Once the spring-damper-mass system becomes more stable, the deformations of the ISIM correspond very good to the results of the paper.

![Graph showing vertical deformation of the beam at midspan under load pattern two of moving spring-damper-mass model with velocity $V = 104.52 \text{ m/s}$, for the DIM and the ISIM compared with the results of the paper for a TBDIM.](image)

**Figure 3.31:** Vertical deformation of the beam at midspan under load pattern two of moving spring-damper-mass model with velocity $V = 104.52 \text{ m/s}$, for the DIM and the ISIM compared with the results of the paper for a TBDIM [17]

<table>
<thead>
<tr>
<th>Method</th>
<th>Calculation time [s]</th>
<th>RRMSE [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIM</td>
<td>583</td>
<td>0.6128</td>
</tr>
<tr>
<td>ISIM</td>
<td>33600</td>
<td>0.3782</td>
</tr>
</tbody>
</table>

**Table 3.14:** Comparison of the two methods with the two-axle vehicle in calculation time and accuracy
3.6 Simulating the methods at non-critical speed

In previous cases, it is considered that the train passes the bridge at resonance speed. In reality, it is avoided to pass a bridge at its critical speed. Therefore, it is important to adjust the traffic speed of trains while passing a Railway bridge to avoid trouble with resonance. Considering the case study of the Temse Bridge, only regional trains cross the bridge. The average speed of a regional train of the NMBS in Belgium is 58.1 $\text{km/h}$ [59]. As discussed in the literature review (2.9), four different vehicle models were compared: the moving constant force model (MCFM), the moving spring-damper-mass model (MSDM), the two-axle vehicle bridge model (TAVBM) and the train-bridge dynamic interaction model (TBDIM). This graph has illustrated the effect of the choice of vehicle model at different speeds for a simply supported beam of 36 m. The simulated results of the four vehicle models differ significantly around resonance speed, but show only slight errors in accuracy for lower speeds. The decrease in complexity of the train models for simulations at lower speeds is allowed. This is of high interest because lower train velocities are closer to the train speeds in reality. Therefore, it is important to compare the accuracy of the models considered in this dissertation at lower train speeds. The simulations of cases B and C are repeated for a speed of 15 $\text{m/s}$, which is close to the average train speed on the Temse bridge.

For case B, the vertical deformation of the beam at midspan under load pattern one is given in figure 3.32. The simulation is compared for both methods: the direct integration method (DIM) and the intersystem iteration method (ISIM). The overall deformation of the beam simulated by both methods is very similar but differs significantly compared with the simulations at resonance speed. As the train is moving at a much lower speed than the resonance speed, the beam would not vibrate anymore around the first bending frequency of the beam. For lower speeds, the more pronounced frequencies in the bridge response are determined by the carriage length $L_c = 26.6$ m.

![Figure 3.32: Vertical deformation of the beam at midspan for case B at a train speed of 15 $\text{m/s}$](image)

As mentioned in the paper of Mao and Lu [60], the response of the bridges is primarily determined by the driving frequency $f_{dr}$ and the dominant frequencies $f_d$ given by equation 3.12 and 3.13, respectively. The driving frequency corresponds to the total duration of the passage of the train and the dominant frequencies are associated with the inter-distance of the repeated loads. The total duration of the passage of the train is $L_b/V = 16.26$ s and gives a very low frequency $f_{dr}$ that represents the global vertical downward deformation during the passage of the train. The vertical deformation of the beam...
contains some higher dominant frequencies, associated with the carriage length \( L_c = 26.6 \text{ m} \). The dominant frequencies are determined by equation 3.12 and summarized in table 3.15 for the first ten modes. The corresponding frequency response of the vertical deformation at midspan determined by both methods is given in figure 3.33 and shows the influence of the different modes \( n \). The first mode \((n = 1)\) is the most predominant for both methods and gives the overall up-and-down movement of the beam during the passage of the train. In the graph of the DITM, an additional higher frequency is markable defined by the seventh mode.

\[
f_{dr} = \frac{nV}{2L_b}
\]

(3.12)

\[
f_{d,n} = \frac{nV}{L_c}
\]

(3.13)

<table>
<thead>
<tr>
<th>Mode number ( n )</th>
<th>( f_{d,n} ) [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.56391</td>
</tr>
<tr>
<td>2</td>
<td>1.1278</td>
</tr>
<tr>
<td>3</td>
<td>1.6917</td>
</tr>
<tr>
<td>4</td>
<td>2.2556</td>
</tr>
<tr>
<td>5</td>
<td>2.8195</td>
</tr>
<tr>
<td>6</td>
<td>3.3835</td>
</tr>
<tr>
<td>7</td>
<td>3.9474</td>
</tr>
<tr>
<td>8</td>
<td>4.5113</td>
</tr>
<tr>
<td>9</td>
<td>5.0752</td>
</tr>
<tr>
<td>10</td>
<td>5.6391</td>
</tr>
</tbody>
</table>

Table 3.15: Summary of the first ten modes with their corresponding dominant frequency \( f_d \)

Figure 3.33: Frequency response curve of the vertical deformation at midspan of the beam under load pattern one, with a train speed of 15 \( \text{m/s} \).
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The redefined results for case C are given in figure 3.34 for a train velocity of 15 m/s. In this case, the train consists of two heavy and shorter locomotives at the begin and end of the train. As a result, the vertical deformation at the beginning and end is significantly larger. Normal passenger cars are between the first and last locomotive, which give a smaller deformations of the bridge at midspan are, similar to case B. The global behavior of the beam gives quitd good results for both methods. As was noticed in case B, several modes of the dominant frequency becomes significant in the DIM.

![Figure 3.34: Vertical deformation at midspan of the beam for case C at train speed of 15 m/s](image)

Finally, the relative root mean square error between both methods is compared and given in table 3.16. The difference in resulted deformation between the DIM and ISIM is 6 to 12 %. This is very low compared to their difference in accuracy at resonance speed, where both methods differ 15 to 25 % from each other. This indicates that both methods gives quite good results at non-critical speed.

<table>
<thead>
<tr>
<th>Simulation</th>
<th>RRMSE [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
<td>0.06859</td>
</tr>
<tr>
<td>Case B</td>
<td>0.1167</td>
</tr>
</tbody>
</table>

Table 3.16: Comparison of the relative root mean square error (RRMSE) between the direct integration method (DIM) and intersystem iteration method (ISIM) for the simulations of case B and C at train speed of 15 m/s

3.6.1 Discussion of the accuracy in Abaqus

During the master dissertation, already some topics according to the accuracy of the results in Abaqus are mentioned. The influence of the mesh size and time increment was discussed for specific cases.

The software Abaqus is a finite element software, meaning that it numerically solves complex differential equations. The finite element method divides the beam into a finite number of small pieces connected to each other by nodes. In general, the finite element model will converge better to the analytic solution for a smaller size of the pieces. By reducing the size of the particles, many more calculations are performed, resulting in a much longer calculation time. Therefore, it is important to find a balance between accuracy and calculation time. For the considered cases of a simply supported beam, the optimum mesh is between 0.2 m and 0.3 m.
The accuracy of the calculation in Abaqus can also be improved by decreasing the size of the time increment. The size of the time increment determines how large the step in time is between two different calculations and regulates the total number of increments. If the size of the time increment becomes too large, successive calculations will diverge and the numerical simulations will be aborted. As decreasing the size of the time increment results in a larger amount of increments, the calculation time will increase. So, also for the time increment, a balance between accuracy and calculation time must be found. The influence of the time increment was already determined for the ISIM, because this method is based on time increments instead of specified time steps like the DIM. The difference between time increments and time steps is shown in figure 3.35, where a time step of 1 s is considered and a time increment of 0.25 s and 0.125 s for respectively time step 2 and 3. The number of time increments needs to be an integer and equals 4 and 8 for respectively step 2 and 3.

Figure 3.35: Visualisation of the terms: time step and time increment

As mentioned before, the optimum size of time increment for the ISIM is 0.01 s. By decreasing the time increment size, the accuracy gain is very small compared to the extra calculation time for the simulation. An increase in time increment gives very bad simulation results and cannot be accepted for further use.

The influence of the time increment on the DIM would have larger effects than expected. For the results simulated at non-critical speed in section 3.6, a time increment size of 0.05 s is considered for the DIM. By decreasing the time increment size in the DIM, several modes of the dominant frequency become important in the resulting vertical deformation of the beam. Figure 3.36 gives the same simulation of case C at the non-critical speed of 15 m/s with an adapted time increment size of 0.01 s. The amplitudes of the DIM increase significantly, indicating the higher modes become more important in the deformation of the beam, which is also visible on the graph.
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Figure 3.36: Vertical deformation at midspan of the beam for case C at train speed $V = 15 \text{ m/s}$, with a time increment size in Abaqus of 0.01 s.

By increasing the time increment size, the opposite happens. Figure 3.37 shows the results for the same simulation of case C at non-critical speed, with a time increment size of 0.1 s. For a large time increment, only the first mode dominates and the DIM approaches the results simulated by the ISIM.

Figure 3.37: Vertical deformation at midspan of the beam for case C at train speed $V = 15 \text{ m/s}$, with a time increment size in Abaqus of 0.1 s.

The global behavior of the beam remains the same, but additional frequencies become dominant by decreasing the time increment. As the vibration of the beam is an important parameter to determine the fatigue of the beam due to the passage of a train, it becomes important to have knowledge about the real behavior to compare both methods. As increasing the time increment size of the ISIM does not results in additional frequencies, it could be concluded that the DIM overestimates the behavior due to the strong coupling between the vehicle and beam.
Chapter 4

Conclusion and future work

This master dissertation discussed two different simulation methods to solve the dynamic wheel-rail interaction, to determine the behavior of the bridge while passing a train. The main goal was to establish a model that simulates the dynamic behavior more accurately than including a dynamic amplification factor in the static calculation as specified in the Eurocode.

First, the importance of a dynamic simulation compared with a static implementation is mentioned. Then, the different ways of modelling a train-bridge interaction system are compared with results found in the literature. It was illustrated that modeling the train as a moving spring-damper-mass model is accurate enough to determine the deflections of the bridge for train speeds far enough from the critical speed. Further, different solution algorithms are discussed where the intersystem iteration method resulted from literature as a computationally more efficient method.

Therefore, the original direct integration method and the more advanced intersystem iteration method were simulated and compared in this study. In chapter 3, the solution methods were explained, and certain assumptions were made to simplify the subsequent simulations. Initially, a simple case of a simply supported beam with a single moving spring-mass model was considered and validated using analytical calculations. Both simulation methods exhibited very good accuracy compared to the analytical results.

The second and third models involved a damped beam loaded with a series of spring-damper-mass models. Due to the increased complexity, it was not feasible to solve the analytical equations, so similar simulations from the literature were used for validation purposes. The validation process yielded excellent results for the intersystem iteration method (ISM), which consistently outperformed the direct integration method (DIM) in terms of accuracy. The main drawback of the DIM was its long calculation time, whereas the ISM offered significantly shorter computation times. Furthermore, the DIM tended to overestimate the vertical deformations at midspan for simulations conducted at resonance speed, making it a less desirable method in such scenarios.

The previous papers reported the results simulated at resonance speed. As was known from the literature, the way of modeling gives significantly different results at resonance speed. Therefore, the simulations are repeated for a lower train velocity of 15 m/s. The results of the direct integration method and intersystem method are noticeably better at lower speeds.

In summary, both methods have their merits depending on the specific application. The direct integration method is recommended as a general first attempt to assess the behavior of a railway bridge under the passage of a train. Meanwhile, the intersystem iteration method, although more computationally intensive, provides more accurate results for predicting the structural health of the railway bridge.

In future work, it would be recommended to validate the simulation methods with real strain measurements on Railway bridges, as for example the case study of the Temse Bridge. This will give a more definite impression of the usefulness of both methods in the engineering world. Since this master
Conclusion and future work

thesis is limited to the vertical wheel-rail interaction for the passage of trains at a constant speed over a bridge, it becomes interesting to study several other cases. How will the horizontal effects affect the results and what would be the influence on the calculation efficiency? Does slowing down/accelerating a train on a railway bridge have a major impact on the deformations of the bridge?
Bibliography


[46] W. Yan and F. D. Fischer, “Applicability of the hertz contact theory to rail-wheel contact problems.”


Appendices
Appendix A: Python code of the intersystem iteration method

This Python script shows the working principle of the intersystem iteration method (ISIM) for the specific case of 32 wheels.

```python
#This python works together with the DLOAD subroutine 'DLOAD_32wheels3.f'.
#It is important to have your cae, odb and subroutine file in the same map as the WorkDirectory,
#so the output of abaqus and the python script must be stored in the same map.
#Further, it is important to have a fixed number of increments in Abaqus, which is the same number in the python script and in the subroutine.

#Import of the needed packages to run a script in Abaqus
from abaqus import *
from abaqusConstants import *
import __main__
import section
import regionToolset
import displayGroupMdbToolset as dgm
import part
import material
import assembly
import step
import interaction
import load
import mesh
import optimization
import job
import sketch
import visualization
import xyPlot
import displayGroupOdbToolset as dgo
import connectorBehavior

#Import of the needed packages to run the python code
import os
import numpy as np
import math

#Defining the parameters during the calculation used in Abaqus: timestep, velocity V, mesh size

timestep = 0.005  #s
V = float(102.78)  #m/s
mesh = 0.2  #m
N_inc = 600  #The maximal number of time increments in Abaqus

#Make an array with the distance between the different wheels in [m]: [first wheel = 0 m, distance between first and second wheel, ..., distance between last en second last wheel]
L = [float(0), float(2.7), float(16.3), float(2.7), float(4.9), float(2.7), float(16.3), float(2.7), float(4.9), float(2.7), float(16.3), float(2.7), float(4.9), float(2.7), float(16.3), float(2.7), float(4.9), float(2.7), float(16.3), float(2.7), float(4.9), float(2.7), float(16.3), float(2.7), float(4.9), float(2.7), float(16.3), float(2.7), float(4.9), float(2.7), float(16.3), float(2.7), float(4.9), float(2.7), float(16.3), float(2.7)]

#Determining the frame step for each wheel when it first touch the beam
Frame_wheel = []
for i in range(0,len(L)):
    Frame_wheel.append(int((sum(L[0:i])/V)/timestep))

#Giving the different jobs a name: job_name = 'BEAM_INPUT_SPRING_{}'.format(i)
```

job_name_beam = 'BEAM_INPUT_SPRING'
job_name_spring = 'SPRING_AMPLITUDE'

#Define the number of wheels
number_wheels = 32

#Making an empty array for each wheel
#deformation will later store the deformations of the beam
defformation = []

#RF stores the reaction forces of the wheels
RF = []

#RF_old will stores the reaction forces of the wheels from the previous loop to check convergence
RF_old = []

for i in range(0, number_wheels):
    deformation.append(())
    RF.append([0])
    RF_old.append([11])  # Set to 11 in the start, 1 more than the convergence check

#Convergence check is done, for a convergence check of 10 N
#the difference between the reaction forces is stored in comparison
comparison = []
convergence_check = 10  # N

for i in range(0, len(RF)):
    #If the difference is smaller than the convergence check: FALSE, else TRUE
    check = abs(RF[i][0]-RF_old[i][0])>convergence_check
    comparison.append(check)

#The start of the iteration loops
loops = 0
while True in comparison:
    loops+=1
    print(loops)

    #The iteration loops can be limited in case convergence is never reached
    if loops > 15:
        break

    #Check of the deformation exist, as for the first loop, a random PSD function is given as track irregularity
    if deformation[0]==():
        # Define the sampling frequency and duration of the signal
        fs = 1  # Hz
        duration = timestep*N_inc  # seconds
        n_samples = N_inc

        # Define the frequency range of the spectrum
        f = np.linspace(0.0001, fs/2, n_samples//2 + 1)

        # Define the parameters of the German Low disturb spectrum
        f0 = 2  # Hz
        A_v = 1.5861e-6  # rad m
        omega = f  # Hz
        omega_R = 0.0206  # rad/m
        omega_C = 0.8246  # rad/m

        # Define the transfer function of the spectrum
        H = (A_v*omega_C**2)/((omega**2 + omega_R**2)*(omega**2+omega_C**2))

        # Generate white noise
        n = np.random.randn(n_samples)
        # Compute the Fourier transform of the noise
        N = np.fft.rfft(n)
# Multiply the Fourier transform of the noise by the transfer function of the spectrum
Y = N * H

# Compute the inverse Fourier transform of the filtered noise
y = np.fft.irfft(Y)
t = np.linspace(0, duration, n_samples)

# Store the deformations of wheel i in deformation[i-1] together with the corresponding time step when the wheel will start to move
for i in range(0, len(t)):
    deformation[0] = deformation[0] + ((i*timestep,y[i]),)
    deformation[1] = deformation[1] + (((i+Frame_wheel[1])*timestep,y[i]),)
    deformation[8] = deformation[8] + (((i+Frame_wheel[8])*timestep,y[i]),)
    deformation[10] = deformation[10] + (((i+Frame_wheel[10])*timestep,y[i]),)
    deformation[12] = deformation[12] + (((i+Frame_wheel[12])*timestep,y[i]),)
    deformation[14] = deformation[14] + (((i+Frame_wheel[14])*timestep,y[i]),)
    deformation[16] = deformation[16] + (((i+Frame_wheel[16])*timestep,y[i]),)
    deformation[17] = deformation[17] + (((i+Frame_wheel[17])*timestep,y[i]),)
    deformation[18] = deformation[18] + (((i+Frame_wheel[18])*timestep,y[i]),)
    deformation[19] = deformation[19] + (((i+Frame_wheel[19])*timestep,y[i]),)
    deformation[21] = deformation[21] + (((i+Frame_wheel[21])*timestep,y[i]),)
    deformation[22] = deformation[22] + (((i+Frame_wheel[22])*timestep,y[i]),)
    deformation[23] = deformation[23] + (((i+Frame_wheel[23])*timestep,y[i]),)
    deformation[26] = deformation[26] + (((i+Frame_wheel[26])*timestep,y[i]),)
    deformation[27] = deformation[27] + (((i+Frame_wheel[27])*timestep,y[i]),)
    deformation[28] = deformation[28] + (((i+Frame_wheel[28])*timestep,y[i]),)
    deformation[29] = deformation[29] + (((i+Frame_wheel[29])*timestep,y[i]),)
deformation[31] = deformation[31] + (((i+Frame_wheel[31])*timestep,y[i]),)

print(deformation[0])

# For steps different from the first step, the deformation is extracted from the beam
else:
    print(deformation[0])

# Import the current deformation data of the beam as amplitude to the spring
mdb.models['SPRING'].TabularAmplitude(name='Wheel1', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[0])
mdb.models['SPRING'].TabularAmplitude(name='Wheel2', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[1])
mdb.models['SPRING'].TabularAmplitude(name='Wheel3', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[2])
mdb.models['SPRING'].TabularAmplitude(name='Wheel4', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[3])
mdb.models['SPRING'].TabularAmplitude(name='Wheel5', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[4])
mdb.models['SPRING'].TabularAmplitude(name='Wheel6', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[5])
mdb.models['SPRING'].TabularAmplitude(name='Wheel7', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[6])
mdb.models['SPRING'].TabularAmplitude(name='Wheel8', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[7])
mdb.models['SPRING'].TabularAmplitude(name='Wheel9', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[8])
mdb.models['SPRING'].TabularAmplitude(name='Wheel10', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[9])
mdb.models['SPRING'].TabularAmplitude(name='Wheel11', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[10])
mdb.models['SPRING'].TabularAmplitude(name='Wheel12', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[11])
mdb.models['SPRING'].TabularAmplitude(name='Wheel13', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[12])
mdb.models['SPRING'].TabularAmplitude(name='Wheel14', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[13])
mdb.models['SPRING'].TabularAmplitude(name='Wheel15', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[14])
mdb.models['SPRING'].TabularAmplitude(name='Wheel16', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[15])
mdb.models['SPRING'].TabularAmplitude(name='Wheel17', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[16])
mdb.models['SPRING'].TabularAmplitude(name='Wheel18', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[17])
mdb.models['SPRING'].TabularAmplitude(name='Wheel19', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[18])
mdb.models['SPRING'].TabularAmplitude(name='Wheel20', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[19])
mdb.models['SPRING'].TabularAmplitude(name='Wheel21', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[20])
mdb.models['SPRING'].TabularAmplitude(name='Wheel22', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[21])
mdb.models['SPRING'].TabularAmplitude(name='Wheel23', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[22])
mdb.models['SPRING'].TabularAmplitude(name='Wheel24', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[23])
mdb.models['SPRING'].TabularAmplitude(name='Wheel25', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[24])
mdb.models['SPRING'].TabularAmplitude(name='Wheel26', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[25])
mdb.models['SPRING'].TabularAmplitude(name='Wheel27', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[26])

mdb.models['SPRING'].TabularAmplitude(name='Wheel28', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[27])

mdb.models['SPRING'].TabularAmplitude(name='Wheel29', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[28])

mdb.models['SPRING'].TabularAmplitude(name='Wheel30', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[29])

mdb.models['SPRING'].TabularAmplitude(name='Wheel31', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[30])

mdb.models['SPRING'].TabularAmplitude(name='Wheel32', timeSpan=STEP, smooth=SOLVER_DEFAULT, data=deformation[31])

#SOLVING THE SPRING SUBSYSTEM
# Make job object of the spring
job_spring = mdb.Job(name=job_name_spring, model=mdb.models['SPRING'], numCpus=8, numDomains=8)

# Submit the spring to the job
job_spring.submit(consistencyChecking=OFF)

# Blocking code -- scripts waits until job is completed
job_spring.waitForCompletion()

# Get the current working directory and open the odb file
odb_spring = session.openOdb(name="{}\{}.odb".format(os.getcwd(), job_name_spring))

# Export the data from the odb file of the spring after submitting
step_spring = odb_spring.steps['Step-1']

# Make list of the vertical reaction force RF2 in function of time: IMPORTANT
# THAT THE FIELD AND HISTORY OUTPUT INCREMETNS = 1 (instead of 10)
for i in range(0, number_wheels):
    RF_old[i] = RF[i]
    RF[i] =[]
    reaction1 = step_spring.historyRegions['Node WHEEL' + str(i+1)+ '-1.1'].historyOutputs['RF2'].data
    RF[i] = np.array([x[1] for x in reaction1])
    print(RF[i])

#In the first loop, the length of both will differ. (Not needed in further
#loops)
if len(RF_old[i]) != len(RF[i]):
    RF_old[i] = np.zeros(len(RF[i]))

#The convergence check is checke again in the loop
comparison = []
for j in range(0, len(RF[i])):
    check = abs(RF[i][j]-RF_old[i][j])>convergence_check
    comparison.append(check)
print(comparison)

#the reaction forces are written to a txt file for each wheel
with open('reaction' +str(i+1) +'.txt', 'w') as f:
    for lines in RF[i]:
        f.write(str(lines))
        f.write('
')

#SOLVING THE SPRING SUBSYSTEM
# Make job object, and load the correct name of the DLOAD subroutine file
job = mdb.Job(name=job_name_beam, model=mdb.models['BEAM'], numCpus=12,
numDomains=12, userSubroutine="{}\{}.f".format(os.getcwd(), 'DLOAD_32wheels3'))
# Submit the beam to the job
job.submit(consistencyChecking=OFF)

# Blocking code -- scripts waits until job is completed
job.waitForCompletion()

# Get the current working directory and open the odb file
odb = session.openOdb(name="{}\{}.odb".format(os.getcwd(), job_name_beam))

# Export the data from the odb file of the beam after submitting
step = odb.steps['Step-1']

# List of [coordinate X, nodelabel], as check if the correct nodes are considered
values2 = step.frames[1].fieldOutputs['U'].values[1].instance.nodes
coX = np.array([[x.coordinates[2], x.label] for x in values2])

# Define the specific nodes that are considered depending on the mesh.
# With Node 1 = 10859, the difference between successive nodes in a same step = 10,
# the total number of nodes along the length = 180
coX1 = np.array([coX[10859-i*10] for i in range(0,180)])
print('coX1 =', coX1)

# Storing the new deformations of the beam
u = []
# defines the location when the wheel first is placed on the beam
location_wheel1 = int((V*timestep)/mesh)
# The deformations are defined at the location of each wheel
for i in range(0, number_wheels):
    u1 = []
    # Iterated over all the frames steps to store the deformation for wheel i
    for frame in step.frames:
        # Stored only from the moment the wheel is on the beam
        if frame.frameId >= Frame_wheel[i]:
            framei = frame.frameId - Frame_wheel[i]
            values = step.frames[frame.frameId].fieldOutputs['U'].values[10859-framei*10*location_wheel1]
            u1.append(values.data[1])
    u.append(u1)

# give the different arrays an equal length
for i in range(0, len(u)):
    while len(u[i]) != len(u[0]):
        u[i].append(0.0)

# store the deformation in the right format again to give as input boundary
# conditions to the wheels
defformation = []
for i in range(0, number_wheels):
    deformation.append(())

for i in range(0, len(u)):  
    deformation[0] = deformation[0] + ((i*timestep,u[0][i]),)
    deformation[1] = deformation[1] + (((i+Frame_wheel[1])*timestep,u[1][i]),)


deflection[8] = deformation[8] + (((i+Frame_wheel[8])*timestep,u[8][i]),)


deformation[12] = deformation[12] + (((i+Frame_wheel[12])*timestep,u[12][i]),)


deformation[16] = deformation[16] + (((i+Frame_wheel[16])*timestep,u[16][i]),)

deflection[17] = deformation[17] + (((i+Frame_wheel[17])*timestep,u[17][i]),)

deformation[18] = deformation[18] + (((i+Frame_wheel[18])*timestep,u[18][i]),)


deformation[22] = deformation[22] + (((i+Frame_wheel[22])*timestep,u[22][i]),)

deflection[23] = deformation[23] + (((i+Frame_wheel[23])*timestep,u[23][i]),)


deformation[26] = deformation[26] + (((i+Frame_wheel[26])*timestep,u[26][i]),)

deflection[27] = deformation[27] + (((i+Frame_wheel[27])*timestep,u[27][i]),)

deformation[28] = deformation[28] + (((i+Frame_wheel[28])*timestep,u[28][i]),)

deflection[29] = deformation[29] + (((i+Frame_wheel[29])*timestep,u[29][i]),)


deflection[31] = deformation[31] + (((i+Frame_wheel[31])*timestep,u[31][i]),)

Listing 1: ISIM Python script for 32 wheels
Appendix B: DLOAD subroutine

The DLOAD subroutine corresponds to the Python script 4, where 32 wheels are considered.

C ONLY EDIT WHERE IT SAYS TO DO SO!!!!
C Subroutine used to read in the external data
SUBROUTINE UEXTERNALDB(LOP,LRESTART,TIME,DTIME,KSTEP,KINC)

C INCLUDE 'ABA_PARAM.INC'
C
DIMENSION TIME(2)

PARAMETER (d = 301) !Put d equal to the maximum number of increments in Abaqus (N_inc)

C Communication block between the 2 subroutines, used to pass down information
DIMENSION W1(d),W2(d),W3(d),W4(d),W5(d),W6(d),W7(d),W8(d),W9(d),W10(d),W11(d),W12(d),W13(d),W14(d),W15(d),W16(d),W17(d),W18(d),W19(d),W20(d),W21(d),W22(d),W23(d),W24(d),W25(d),W26(d),W27(d),W28(d),W29(d),W30(d),W31(d),W32(d)

C EDIT: the filepath to the correct ones, namely the path of the workdirectory in Abaqus
IF(LOP==0)THEN

!The reaction forces in time for different wheels
OPEN(UNIT=15, FILE="D:\SD\Series_MSDM\32_wheels\reaction1.txt") !wheel11
OPEN(UNIT=16, FILE="D:\SD\Series_MSDM\32_wheels\reaction2.txt") !wheel12
OPEN(UNIT=17, FILE="D:\SD\Series_MSDM\32_wheels\reaction3.txt") !wheel13
OPEN(UNIT=18, FILE="D:\SD\Series_MSDM\32_wheels\reaction4.txt") !wheel14
OPEN(UNIT=19, FILE="D:\SD\Series_MSDM\32_wheels\reaction5.txt") !wheel15
OPEN(UNIT=20, FILE="D:\SD\Series_MSDM\32_wheels\reaction6.txt") !wheel16
OPEN(UNIT=21, FILE="D:\SD\Series_MSDM\32_wheels\reaction7.txt") !wheel17
OPEN(UNIT=22, FILE="D:\SD\Series_MSDM\32_wheels\reaction8.txt") !wheel18
OPEN(UNIT=23, FILE="D:\SD\Series_MSDM\32_wheels\reaction9.txt") !wheel19
OPEN(UNIT=24, FILE="D:\SD\Series_MSDM\32_wheels\reaction10.txt") !wheel10
OPEN(UNIT=25, FILE="D:\SD\Series_MSDM\32_wheels\reaction11.txt")
OPEN(UNIT=26, FILE="D:\SD\Series_MSDM\32_wheels\reaction12.txt")
OPEN(UNIT=27, FILE="D:\SD\Series_MSDM\32_wheels\reaction13.txt")
OPEN(UNIT=28, FILE="D:\SD\Series_MSDM\32_wheels\reaction14.txt")
OPEN(UNIT=29, FILE="D:\SD\Series_MSDM\32_wheels\reaction15.txt")
OPEN(UNIT=30, FILE="D:\SD\Series_MSDM\32_wheels\reaction16.txt")
OPEN(UNIT=31, FILE="D:\SD\Series_MSDM\32_wheels\reaction17.txt")
OPEN(UNIT=32, FILE="D:\SD\Series_MSDM\32_wheels\reaction18.txt")
OPEN(UNIT=33, FILE="D:\SD\Series_MSDM\32_wheels\reaction19.txt")
OPEN(UNIT=34, FILE="D:\SD\Series_MSDM\32_wheels\reaction20.txt")
OPEN(UNIT=35, FILE="D:\SD\Series_MSDM\32_wheels\reaction21.txt")
OPEN(UNIT=36, FILE="D:\SD\Series_MSDM\32_wheels\reaction22.txt")
OPEN(UNIT=37, FILE="D:\SD\Series_MSDM\32_wheels\reaction23.txt")
OPEN(UNIT=38, FILE="D:\SD\Series_MSDM\32_wheels\reaction24.txt")
OPEN(UNIT=39, FILE="D:\SD\Series_MSDM\32_wheels\reaction25.txt")
OPEN(UNIT=40, FILE="D:\SD\Series_MSDM\32_wheels\reaction26.txt")
OPEN(UNIT=41, FILE="D:\SD\Series_MSDM\32_wheels\reaction27.txt")
OPEN(UNIT=42, FILE="D:\SD\Series_MSDM\32_wheels\reaction28.txt")
OPEN(UNIT=43, FILE="D:\SD\Series_MSDM\32_wheels\reaction29.txt")
OPEN(UNIT=44, FILE="D:\SD\Series_MSDM\32_wheels\reaction30.txt")
OPEN(UNIT=45, FILE="D:\SD\Series_MSDM\32_wheels\reaction31.txt")
OPEN(UNIT=46, FILE="D:\SD\Series_MSDM\32_wheels\reaction32.txt")
Read the file with reaction forces data of wheel i and stores to the corresponding variable wi

Read(15,*)W1
Read(16,*)W2
Read(17,*)W3
Read(18,*)W4
Read(19,*)W5
Read(20,*)W6
Read(21,*)W7
Read(22,*)W8
Read(23,*)W9
Read(24,*)W10
Read(25,*)W11
Read(26,*)W12
Read(27,*)W13
Read(28,*)W14
Read(29,*)W15
Read(30,*)W16
Read(31,*)W17
Read(32,*)W18
Read(33,*)W19
Read(34,*)W20
Read(35,*)W21
Read(36,*)W22
Read(37,*)W23
Read(38,*)W24
Read(39,*)W25
Read(40,*)W26
Read(41,*)W27
Read(42,*)W28
Read(43,*)W29
Read(44,*)W30
Read(45,*)W31
Read(46,*)W32

Elseif (LOP == 3) then

Closes the files with reaction force data

Close(15)
Close(16)
Close(17)
Close(18)
Close(19)
Close(20)
Close(21)
Close(22)
Close(23)
Close(24)
Close(25)
Close(26)
Close(27)
Close(28)
Close(29)
Close(30)
Close(31)
Close(32)
Close(33)
Close(34)
Close(35)
Close(36)
Close(37)
Close(38)
Close(39)
Close(40)
Close(41)
C Subroutine used to describe to load on the desired position on the beam
SUBROUTINE DLOAD(F,KSTEP,KINC,TIME,NOEL,NPT,LAYER,KSPT,COORDS,JI,NAME)
C
INCLUDE 'ABA_PARAM.INC'
C
PARAMETER (d = 301) !Put d equal to the maximum number of increments in Abaqus (N_inc)
DIMENSION TIME(2), COORDS(3)
CHARACTER*80 SNAME

C Communication block between the 2 subroutines, used to pass down information
DIMENSION W1(d),W2(d),W3(d),W4(d),W5(d),W6(d),W7(d),W8(d),W9(d),W10(d),W11(d),
W12(d),W13(d),W14(d),W15(d),W16(d),W17(d),W18(d),W19(d),W20(d),W21(d),W22(d),W23
(d),W24(d),W25(d),W26(d),W27(d),W28(d),W29(d),W30(d),W31(d),W32

C Defining coordinates X,Y,Z and the timestep T (see in Abaqus how they are oriented
, this is user and model dependent)
X=COORDS(1) !lenght correlates to lenght of the beam
Y=COORDS(2) !height of the beam
Z=COORDS(3) !width of the beam
T=TIME(1) !Time step of the calculations

C EDIT: Properties of the train and specify the location of the wheels
V = 102.78 !Speed of the vehicle m/s
w = 1.8 !Width of the beam
h = 1.8 !Height of the beam
Tstep = 0.01 !Size of the time increment in Abaqus
B_w = w/2 !Define the location of the wheel in the transverse direction of
the beam, for this case in the middle

!Define the locations of the different wheels, depending on the distance between
the wheels
!Location of the first set of wheels
X0= 0 !distance from the begin of the beam where the train starts
XL= 0.2 !contact area of the wheel, equals the mesh size if a point load is
considered
Xcenter=X0+V*T !Determines the location of the train that varies in function of
the speed and timesteps

!Location of the second set of wheels
X2 = -2.7
Xcenter2 = X2+V*T
wheel2 = INT(ABS(X2)/(V*Tstep))

!Location third wheel
X3 = X2-16.3
Xcenter3 = X3+V*T
wheel3 = INT(ABS(X3)/(V*Tstep))
!location fourth wheel
X4 = X3-2.7
Xcenter4 = X4+V’T
wheel4 = INT(ABS(X4)/(V’Tstep))

!location wheel5
X5 = X4-4.9
Xcenter5 = X5+V’T
wheel5 = INT(ABS(X5)/(V’Tstep))

!location wheel6
X6 = X5-2.7
Xcenter6 = X6+V’T
wheel6 = INT(ABS(X6)/(V’Tstep))

!location wheel7
X7 = X6-16.3
Xcenter7 = X7+V’T
wheel7 = INT(ABS(X7)/(V’Tstep))

!location wheel8
X8 = X7-2.7
Xcenter8 = X8+V’T
wheel8 = INT(ABS(X8)/(V’Tstep))

!location wheel9
X9 = X8-4.9
Xcenter9 = X9+V’T
wheel9 = INT(ABS(X9)/(V’Tstep))

!location wheel10
X10 = X9-2.7
Xcenter10 = X10+V’T
wheel10 = INT(ABS(X10)/(V’Tstep))

!location wheel11
X11=X10-16.3
Xcenter11=X11+V’T
wheel11 = INT(ABS(X11)/(V’Tstep))

!location wheel12
X12=X11-2.7
Xcenter12=X12+V’T
wheel12 = INT(ABS(X12)/(V’Tstep))

!location wheel13
X13=X12-4.9
Xcenter13=X13+V’T
wheel13 = INT(ABS(X13)/(V’Tstep))

!location wheel14
X14=X13-2.7
Xcenter14=X14+V’T
wheel14 = INT(ABS(X14)/(V’Tstep))

!location wheel15
X15=X14-16.3
Xcenter15=X15+V’T
wheel15 = INT(ABS(X15)/(V’Tstep))

!location wheel16
X16=X15-2.7
Xcenter16 = X16 + V*T
wheel16 = INT(ABS(X16)/(V*Tstep))

!location wheel17
X17 = X16 - 4.9
Xcenter17 = X17 + V*T
wheel17 = INT(ABS(X17)/(V*Tstep))

!location wheel18
X18 = X17 - 2.7
Xcenter18 = X18 + V*T
wheel18 = INT(ABS(X18)/(V*Tstep))

!location wheel19
X19 = X18 - 16.3
Xcenter19 = X19 + V*T
wheel19 = INT(ABS(X19)/(V*Tstep))

!location wheel20
X20 = X19 - 2.7
Xcenter20 = X20 + V*T
wheel20 = INT(ABS(X20)/(V*Tstep))

!location wheel21
X21 = X20 - 4.9
Xcenter21 = X21 + V*T
wheel21 = INT(ABS(X21)/(V*Tstep))

!location wheel22
X22 = X21 - 2.7
Xcenter22 = X22 + V*T
wheel22 = INT(ABS(X22)/(V*Tstep))

!location wheel23
X23 = X22 - 16.3
Xcenter23 = X23 + V*T
wheel23 = INT(ABS(X23)/(V*Tstep))

!location wheel24
X24 = X23 - 2.7
Xcenter24 = X24 + V*T
wheel24 = INT(ABS(X24)/(V*Tstep))

!location wheel25
X25 = X24 - 4.9
Xcenter25 = X25 + V*T
wheel25 = INT(ABS(X25)/(V*Tstep))

!location wheel26
X26 = X25 - 2.7
Xcenter26 = X26 + V*T
wheel26 = INT(ABS(X26)/(V*Tstep))

!location wheel27
X27 = X26 - 16.3
Xcenter27 = X27 + V*T
wheel27 = INT(ABS(X27)/(V*Tstep))

!location wheel28
X28 = X27 - 2.7
Xcenter28 = X28 + V*T
wheel28 = INT(ABS(X28)/(V*Tstep))
!location wheel29
X29=X28-4.9
Xcenter29=X29+V*T
wheel29= INT(ABS(X29)/(V*Tstep))

!location wheel30
X30=X29-2.7
Xcenter30=X30+V*T
wheel30= INT(ABS(X30)/(V*Tstep))

!location wheel31
X31=X30-16.3
Xcenter31=X31+V*T
wheel31= INT(ABS(X31)/(V*Tstep))

!location wheel32
X32=X31-2.7
Xcenter32=X32+V*T
wheel32= INT(ABS(X32)/(V*Tstep))

!Determine the location of the wheel and the corresponding wheel-rail interaction force
i = 0 !Determines every step
F = 0 !The force is zero at the locations different from the contact points between the wheel and beam

C To debug, you can write variables to the dat file
C WRITE(6,*) 'REACTION FORCE',B
WRITE(6,*) 'Other Vars',wheel2, wheel15, wheel32

C Code to determine the location of each wheel on the beam.
do while (i.le.d)
  if (KINC.eq.i) then
    if (Z>(Xcenter-XL/2) .and. Z<(Xcenter+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
      F = W1(i)/(XL*XL)
    else if (Z>(Xcenter2-XL/2) .and. Z<(Xcenter2+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
      F = W2(i)/(XL*XL)
    else if (Z>(Xcenter3-XL/2) .and. Z<(Xcenter3+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
      F = W3(i)/(XL*XL)
    else if (Z>(Xcenter4-XL/2) .and. Z<(Xcenter4+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
      F = W4(i)/(XL*XL)
    else if (Z>(Xcenter5-XL/2) .and. Z<(Xcenter5+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
      F = W5(i)/(XL*XL)
    else if (Z>(Xcenter6-XL/2) .and. Z<(Xcenter6+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
      F = W6(i)/(XL*XL)
    else if (Z>(Xcenter7-XL/2) .and. Z<(Xcenter7+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
      F = W7(i)/(XL*XL)
    else if (Z>(Xcenter8-XL/2) .and. Z<(Xcenter8+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
      F = W8(i)/(XL*XL)
    else if (Z>(Xcenter9-XL/2) .and. Z<(Xcenter9+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
      F = W9(i)/(XL*XL)
  else if (Z>(Xcenter10-XL/2) .and. Z<(Xcenter10+XL/2) .and. X>(B_w-XL/2) .and. X<(B_w+XL/2)) then
    F = W10(i)/(XL*XL)
end if
\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center11} - X_L/2) \text{ and } Z < (X_{center11} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel11}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center12} - X_L/2) \text{ and } Z < (X_{center12} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel12}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center13} - X_L/2) \text{ and } Z < (X_{center13} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel13}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center14} - X_L/2) \text{ and } Z < (X_{center14} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel14}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center15} - X_L/2) \text{ and } Z < (X_{center15} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel15}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center16} - X_L/2) \text{ and } Z < (X_{center16} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel16}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center17} - X_L/2) \text{ and } Z < (X_{center17} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel17}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center18} - X_L/2) \text{ and } Z < (X_{center18} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel18}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center19} - X_L/2) \text{ and } Z < (X_{center19} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel19}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center20} - X_L/2) \text{ and } Z < (X_{center20} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel20}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center21} - X_L/2) \text{ and } Z < (X_{center21} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel21}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center22} - X_L/2) \text{ and } Z < (X_{center22} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel22}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center23} - X_L/2) \text{ and } Z < (X_{center23} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel23}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center24} - X_L/2) \text{ and } Z < (X_{center24} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel24}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center25} - X_L/2) \text{ and } Z < (X_{center25} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel25}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center26} - X_L/2) \text{ and } Z < (X_{center26} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel26}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center27} - X_L/2) \text{ and } Z < (X_{center27} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel27}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center28} - X_L/2) \text{ and } Z < (X_{center28} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel28}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center29} - X_L/2) \text{ and } Z < (X_{center29} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel29}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center30} - X_L/2) \text{ and } Z < (X_{center30} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel30}) \text{then} \]

\[ F = \frac{W_1(i)}{(X_L^2)} \]

\[ \text{else if } (Z > (X_{center31} - X_L/2) \text{ and } Z < (X_{center31} + X_L/2) \text{ and } X > (B_w - X_L/2) \text{ and } X < (B_w + X_L/2) \text{ and } KINC > \text{wheel31}) \text{then} \]
\[
F = \frac{W31(i)}{XL^2}
\]

\[
\text{else if } (Z > (Xcenter32 - XL/2) \text{ and } Z < (Xcenter32 + XL/2) \text{ and } X > (B_w - XL/2) \text{ and } X < (B_w + XL/2) \text{ and } KINC > \text{wheel32}) \text{ then}
\]
\[
F = \frac{W32(i)}{XL^2}
\]

\[
\text{else}
\]
\[
F = 0 \text{ !At all the other locations, the wheel is not acting there}
\]

\[
\text{end if}
\]

\[
i = i + 1
\]

\[
\text{end do}
\]

\[
\text{RETURN}
\]

\[
\text{END}
\]