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Cospectrality results on generalized Johnson and Grassmann graphs

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Preface

This master's thesis would not have come to a successful conclusion without the excellent help of my two supervisors, dr. Aida Abiad and dr. Jozefien D'haeseleer. They were always available when I had questions and provided me with many interesting problems to tackle. Thank you Jozefien, for your contagious passion and enthusiasm for finite geometry. Thank you Aida, for the great amount of references you offered me, and for giving me opportunities to present my work in a professional academic context.

I would like to address a special word of appreciation to prof. dr. Leo Storme. Thank you for your time and effort in reading this thesis. But above all, thank you for inspiring me in the topic of discrete mathematics during my academic years.

Furthermore, I want to thank dr. Jens Bossaert for providing the LaTeX-template that I used for the title page. I also want to thank Arne Botteldoorn and Steven Van Overberghe for helping me optimize my Python code. Without them, I would not have found as many new results as I have.

Finally, a word of thanks to my family and friends, for being there when I need them the most.

Robin Simoens, May 31, 2022

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Contents

Conventions

Abbreviation	Meaning	Page
DS	determined by its spectrum	16
GM-switching	Godsil-McKay switching	20
NDS	not determined by its spectrum	16
WQH-switching	Wang-Qiu-Hu switching	22

Structure	Notation
natural number	$a, b, c, d, h, i, j, k, l, m, n, s, \lambda, \mu$
prime number	p
prime power	q
set	C, D, E, S, V
graph	Γ
vertex	u, v, w
vector	$\vec{x}, \vec{y}, \vec{z}$
matrix	$A, B, D, L, M, N, P, Q, S$
field	$\mathbb{F}, \mathbb{K}$
vector space	V
point	p, q
line	L, M
point-line geometry: order parameters	s, t
point-line geometry	$\mathcal{S}$
point set, line set, incidence relation	$\mathcal{P}, \mathcal{L}, \mathcal{I}$

Notation	**Meaning**	**Page**
$\alpha(\Gamma)$	independence number of the graph Γ	6
$A \triangle B$	symmetric difference of the sets A and B	18
C_n	cycle graph on n vertices	9
$\mathrm{diam}(\Gamma)$	diameter of the graph Γ	6
$\mathrm{d}(v, w)$	distance between the vertices v and w	6
$E(\Gamma)$	edge set of the graph Γ	5
$\mathcal{E}(\Gamma)$	energy of the graph Γ	2
$\mathbb{F}_q$	finite field of order q	10
g_n	number of nonisomorphic graphs on n vertices	71
$\mathrm{g}(\Gamma)$	girth of the graph Γ	6
$\overline{\Gamma}$	complement of the graph Γ	5
$\Gamma \cup \Gamma'$	disjoint union of the graphs Γ and Γ'	16
I or I_n	$(n \times n)$ identity matrix	9
J or J_n	$(n \times n)$ all-one matrix	9
$J(n, k)$	Johnson graph with parameters n and k	27
$J_q(n, k)$	Grassmann or q-Johnson graph with parameters q, n and k	31
$J_S(n, k)$	generalized Johnson graph with parameters S, n and k	29
$J_{q,S}(n, k)$	generalized Grassmann graph with parameters q, S, n and k	32
K_n	complete graph on n vertices	5
$\overline{K_n}$	edgeless graph on n vertices	5
$K_{m,n}$	complete bipartite graph with bipartition classes of size m and n	17
$K(n, k)$	Kneser graph with parameters n and k	26
$K_q(n, k)$	q-Kneser graph with parameters q, n and k	30
$\lambda(v, w)$	common neighbour count of the vertices v and w	55
$\Lambda(v)$	common neighbour pattern of the vertex v	55
$\binom{n}{k}$	binomial coefficient	12
$\begin{bmatrix} n \\ k \end{bmatrix}_q$	Gaussian or q-binomial coefficient	11
O_n	Odd graph on $\binom{2n-1}{n-1}$ vertices	27
$o(f(n))$	quantity that is asymptotically smaller than every multiple of $f(n)$	68
$O(f(n))$	quantity that is asymptotically smaller than a multiple of $f(n)$	2
P_n	path graph on n vertices	17
$\mathrm{PG}(V)$	projective space coming from the vector space V	10
$\mathrm{PG}(n-1, \mathbb{K})$	projective space coming from the vector space $\mathbb{K}^n$	10
$\mathrm{PG}(n-1, q)$	projective space coming from the vector space $\mathbb{F}_q^n$	10
$\mathrm{PG}_W(V)$	residual projective space of the subspace W in $\mathrm{PG}(V)$	13
$S + n$	$\{s + n \mid s \in S\}$	29
T_n	triangular graph on $\binom{n}{2}$ vertices	28
$V(\Gamma)$	vertex set of the graph Γ	5
$v \sim w$	the vertices v and w are adjacent	5
$v \not\sim w$	the vertices v and w are nonadjacent	5
$\chi(\Gamma)$	chromatic number of the graph Γ	6
$\omega(\Gamma)$	clique number of the graph Γ	6

Summary of new results

The following list contains all new findings that are included in this thesis.

- New result 4.10: the three graphs $J_{\{1\}}(11, 4)$, $J_{\{2,4\}}(10, 5)$ and $K_3(4, 2)$ are not determined by their spectrum (page 38).
- Tables 4.1 to 4.7: improved numbers for the nonexistence of certain sizes of GM- and WQH-switching sets in small generalized Johnson and Grassmann graphs (pages 39–41).
- Section 5.4: observations that the proofs of Theorem 5.5 and Theorem 5.11 are not extendable to other graphs in a natural way (page 49).
- Section 6.3: observations that the proofs of Theorem 6.1 and Theorem 6.3 are not extendable to their q-analogon in a natural way (pages 52–53).
- Theorem 6.4: a lower bound on the size of a GM- or WQH-switching set in certain generalized Grassmann graphs (on page 53).
- Corollary 8.5: the generalized Johnson graph $J_{\{2\}}(n, 4)$ is not determined by its spectrum if $n \geq 8$ (page 60).
- Corollary 9.4: the q-Kneser graph $K_2(n, k)$ is not determined by its spectrum (page 62).
- Theorem 10.4 and Theorem 10.5: explicit expressions for the diameter and girth of generalized Grassmann graphs in terms of their parameters (page 65).
- Lemma 11.5, Lemma 11.6 and Theorem 11.7: an elaborate proof of an asymptotic lower bound on the number of graphs that are not determined by their spectrum (pages 69–72). However, the statement is not new. It can be found in [42], along with a short sketch of the proof.
- Appendix C: Python code for enumerating all possible GM- and WQH-switching sets of a given size in a given generalized Johnson or generalized Grassmann graph (pages 83–94).

Introduction

Imagine you are a detective investigating a jewelry theft. You find several footprints at the crime scene. These imprints indicate how heavy and how large the thief is, and his or her walking pattern during the crime. So the footprints tell you much about the behaviour of the culprit. Unfortunately, there are still two possible suspects with those same characteristics. Their names are $C_4 \cup K_1$ and $K_{1,4}$.

Figure 1. $C_4 \cup K_1$ (left) and $K_{1,4}$ (right) have the same spectrum $\{-2, 0, 0, 0, 2\}$.

Given a graph, we can attach to it an array of real numbers: the spectrum. The spectrum contains much information on the graph, like the number of vertices, the number of edges, or the amount of closed walks of fixed length. However, in general, it does not determine a graph completely. So in a way, the spectrum acts like the footprints of our thief. Spectral graph theorists are the detectives investigating the footprints. Though we can derive many structural properties of the graph from its spectrum, there are still graphs with the same spectrum.

In 2003, van Dam and Haemers conjectured that *almost all* graphs are determined by their spectrum [26]. In other words, the proportion of graphs on n vertices that are determined by their spectrum, is expected to approach one as $n \to \infty$. This conjecture gained some numerical and theoretical evidence along the way, but today still, it remains an open problem.

The conjecture is of particular interest in complexity theory, since it is still undecided whether graph isomorphism is a hard or an easy problem. Calculating the spectrum of a graph can happen in polynomial time, so if the amount of cospectral mates remains small, the spectrum could be an effective tool for determining isomorphism.

The computational problem of determining graph isomorphism is not known to be solvable in polynomial time in a deterministic way (P). Though it is solvable in polynomial time in a nondeterministic way (NP), we still do not know if it is NP-complete. Because of its large importance, mathematicians defined a new class (GI) of problems that have a reduction to the graph isomorphism problem in polynomial time. It has also been proved to be equivalent to several similar problems, like the reduced isomorphism problem for regular graphs [64] or determining whether a graph is self-complementary [20].

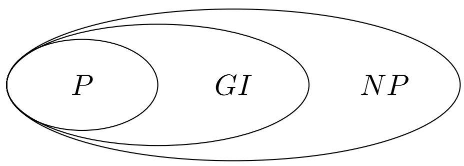


Figure 2. Is the graph isomorphism problem solvable in polynomial time? Is it NP-complete?

In 2015, Babai showed that the graph isomorphism problem is actually closer to being P than the more difficult NP. He announced a theoretically fast, quasipolynomial algorithm that could handle the task in the running time $2^{O(\log(n)^3)}$ [7]. In the corresponding paper, he shows that, in a certain sense, the Johnson graphs $J(n, k)$ are the largest obstructions for effective isomorphism methods. As we shall see, the Johnson graphs are not determined by their spectrum if $3 \leq k \leq n - 3$, which means that calculating their spectrum is not enough to determine isomorphism. This provides extra motivation to investigate graphs with the same spectrum as the Johnson graphs.

The spectral characterization problem also relates to chemistry, where the problem first originated [39]. In the Hückel molecular orbital (HMO) theory, the total π-electron energy of a conjugated molecule is given by

$$E_\pi = \alpha n_e + \beta \sum_{i=1}^{n} |\lambda_i|$$

where α and β are certain standard parameters, n_e is the number of electrons in a π orbital and λ_i are the eigenvalues of the molecular graph (the graph with the atoms as vertices and where two vertices are adjacent if the corresponding atoms are chemically bonded) [40].

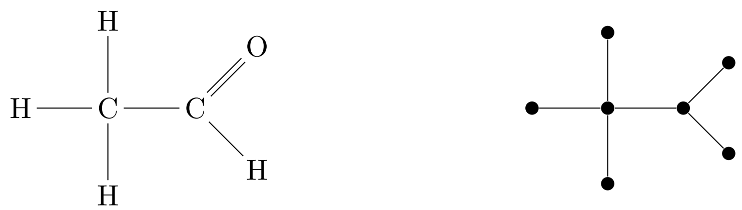


Figure 3. The molecule acetaldehyde (ethanal) and its molecular graph.

This led to the introduction of the notion of *graph energy*, which is defined as the sum

$$\mathcal{E}(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$$

of the absolute values of the eigenvalues of the graph. A graph on n vertices is called *hyperenergetic* if its energy is strictly larger than the energy of K_n, in other words, if $\mathcal{E}(\Gamma) > 2n - 2$. Gutman conjectured that $\mathcal{E}(\Gamma) \leq 2n - 2$ for all graphs, but this proved to be wrong [40]. In fact, any nontrivial Kneser graph $K(n, k)$ with $k \geq 4$ is an example of a hyperenergetic graph [5].

Spectral characterization has applications in many other fields, such as shape analysis, where cospectral mates correspond to different shapes of membranes that produce the same sound when struck [26].

A lot of interest in graph theory goes to strongly regular graphs, which are characterized by having exactly three eigenvalues. The role of strongly regular graphs in graph theory can be compared to that of groups in algebra: it would be very nice if there were a classification of them – but this is not known. There are still many open problems. An important question is to ask how many strongly regular graphs there exist with the same given parameters. This question relates directly to the spectral characterization problem, since cospectral strongly regular graphs have the same parameters and vice versa. More generally, distance-regular graphs are cospectral if and only if they have the same intersection numbers. Therefore, constructing new strongly regular graphs through spectral techniques has received much attention [1, 3, 13, 47, 48, 49].

Most known methods for constructing cospectral graphs work particularly well on graphs coming from finite geometries, since they posses many symmetries. For this reason, the current study started with the following main objective: investigating whether the q-Kneser graphs are determined by their spectrum. The goal was to investigate whether the known results for Kneser graphs [41] extend to q-Kneser graphs. While studying these two families of graphs, it is almost inevitable to stumble upon the Johnson and Grassmann graphs as well, such that, eventually, the interest moved to the overarching classes of generalized Johnson and Grassmann graphs. The research question eventually became the following:

Which generalized Johnson and Grassmann graphs are determined by their spectrum?

To tackle the problem of determining the cospectrality of a graph, one can either prove that the spectrum determines the graph, or one can construct nonisomorphic graphs with the same spectrum. Since the former is in general a difficult task, we focus on the latter, mostly by using *switching techniques*: methods to alter a graph with respect to a given subgraph (the *switching set*) such that the obtained graph has the same spectrum. The methodology can be summarised in the following three steps.

1. Like many mathematical problems, it is often wise to start by looking at some small examples. Therefore, we begin by implementing some small generalized Johnson and Grassmann graphs in a computer program. In this way, a computer can iterate all possible switching sets up to a given size. For these switching sets, it is then checked whether the graph obtained by switching is isomorphic.

2. If the obtained graph is not isomorphic to the original one, then the original graph is not determined by its spectrum. The second step is to make this result formal, often by interpreting the switching set in a geometrical way. At the same time, an attempt is made to embed the result into a bigger (preferably infinite) collection of graphs that are not determined by their spectrum.

3. In order to prove that the graph obtained by switching is not isomorphic to the original one, we test certain graph invariants (such as the clique number or diameter) for being different before and after the switching process. These are our "certificates" of nonisomorphism. This last step is often the most difficult, because one has to make extensive use of what the graph looks like.

We end this introduction with an overview of what the reader can expect from this work. For an English summary, see Appendix B. For a summary in Dutch, see Appendix A.

In **Part I**, we provide all the necessary background on the topic. In **Chapter 1**, some basic notions in graph theory, projective geometry and incidence geometry are covered. **Chapter 2** introduces cospectral graphs and how to construct them. The generalized Johnson and Grassmann graphs are introduced in **Chapter 3**.

The core part of this thesis is **Part II**, which is about all the previous and new results on the cospectrality of generalized Johnson and Grassmann graphs. It contains an overview of all known results in **Chapter 4**, while the other chapters, **Chapter 5** to **Chapter 9**, each treat an individual cospectrality result on one or more families of these graphs. We discuss these results, while pausing here and there to look at possible generalizations (or arguments against them) and offer a few new observations.

Although the emphasis of this thesis lies on cospectrality results, two additional related topics are discussed in **Part III**. In **Chapter 10**, we provide an explicit expression for the diameter and girth of the generalized Johnson and Grassmann graphs. In **Chapter 11**, we determine a lower bound on the number of graphs that are not determined by their spectrum.

This master's thesis ends in **Part IV** (**Chapter 12**) with a conclusion and some open questions for future research.

Part I

Preliminaries

Chapter 1

Definitions and properties

Before we can dive into the subject of generalized Johnson graphs, generalized Grassmann graphs and their cospectrality, we need some background on graph theory and projective geometry. In this chapter, we introduce the necessary definitions and properties from those two areas of mathematics. We also consider point-line geometries, since those will come in handy later on. We suppose that the reader is familiar with basic concepts in combinatorics and linear algebra.

1.1 Graph theory

We start with an overview of some basic concepts in graph theory. We refer to [38] for a more thorough background.

Definition 1.1

A **graph** is a tuple $\Gamma = (V, E)$, where V is a finite set of elements called **vertices**, and E is a set of pairs $\{v, w\}$ with $v, w \in V$, $v \neq w$, called **edges**. We write $V = V(\Gamma)$ and $E = E(\Gamma)$.

Our definition of a graph excludes directed edges, multi-edges and loops. In other words, we only consider simple graphs. We also assume them to be finite.

Graphs are often defined by their vertex set and an adjacency relation:

Definition 1.2

Let Γ be a graph. Two vertices v and w are **adjacent** (notation: $v \sim w$) if $\{v, w\} \in E(\Gamma)$. Adjacent vertices are called **neighbours**. The **degree** $\mathbf{deg}(v)$ of a vertex v is the number of its neighbours.

A graph is **complete** if all vertices are mutually adjacent. A graph is **edgeless** if no two vertices are adjacent.

The **complement** $\overline{\boldsymbol{\Gamma}}$ of Γ is the graph with the same vertex set as Γ, but where vertices are adjacent if and only if they are not adjacent in Γ.

The complete graph on n vertices is denoted by K_n. The edgeless graph on n vertices is its complement and therefore denoted by $\overline{K_n}$.

We present graphs by a figure with dots as vertices and lines between them as edges. A red dashed line between vertices is sometimes used to make clear that they are not adjacent.

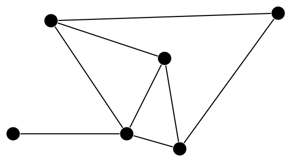


Figure 1.1. A graph.

We will need the following concept of graph isomorphism to equate graphs that look the same.

Definition 1.3

Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be two graphs. An **isomorphism** from Γ to Γ' is a bijection from V to V' that preserves adjacency. Graphs are **isomorphic** if there exists an isomorphism between them. An **automorphism** of Γ is an isomorphism from Γ to itself.

An isomorphism is essentially a relabelling of the vertices of a graph. We will often say that graphs are "equal" while we actually mean "isomorphic".

Definition 1.4

Let $\Gamma = (V, E)$ be a graph. A **clique** is a set $C \subseteq V$ of which every two vertices are adjacent. An **independent set** or **coclique** is a set $C \subseteq V$ of which no two vertices are adjacent. A (co)clique is **maximal** if it is not included in an other (co)clique. A (co)clique is **maximum** if there is no larger (co)clique.

The **clique number** $\boldsymbol{\omega(\Gamma)}$ of Γ is the size of a maximum clique. The **independence number** $\boldsymbol{\alpha(\Gamma)}$ of Γ is the size of a maximum independent set.

In other words, a clique is a complete subgraph and an independent set is an edgeless subgraph. Mind the difference between maximal and maximum cliques. Maximum cliques are always maximal, but the converse is not always true.

Definition 1.5

Let Γ be a graph and $v, w \in V(\Gamma)$. A **walk** (of **length** n) from v to w is a sequence of vertices $(v_0, v_1, \ldots, v_n)$ such that $v_0 = v, \quad v_n = w$ and $v_i \sim v_{i+1}$ for all $i \in \{0, 1, \ldots, n-1\}$. A graph is **connected** if there exists a walk between every pair of vertices.

A walk is **closed** if the first and the last vertex are the same. A **cycle** is a closed walk of length at least three that does not contain the same vertex twice.

The **distance** $\mathbf{d}(\boldsymbol{v}, \boldsymbol{w})$ between two vertices v and w is the length of a shortest walk from v to w, or ∞ if such a walk does not exist.

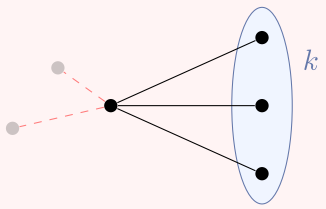
The **diameter** $\mathbf{diam}(\boldsymbol{\Gamma})$ of Γ is the largest possible distance between two vertices of Γ. The **girth** $\mathbf{g}(\boldsymbol{\Gamma})$ is the length of a shortest cycle in the graph, or ∞ if the graph does not contain a cycle.

Most graphs that will be considered later on, are well balanced and feature many symmetries. We make this formal by the following notions of regularity.

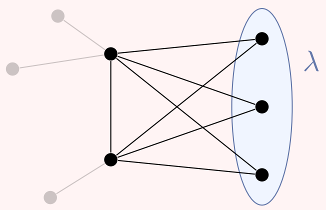
Definition 1.6

Let Γ be a graph on n vertices.

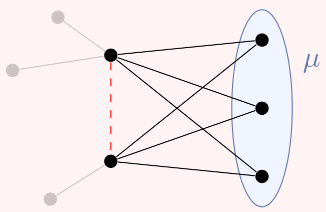
(i) Γ is **regular** if every vertex has the same degree k.



(ii) Γ is **edge-regular** if Γ is regular, not edgeless and every two adjacent vertices have the same number λ of common neighbours.



(iii) Γ is **co-edge-regular** if Γ is regular, not complete and every two nonadjacent vertices have the same number μ of common neighbours.



(iv) Γ is **strongly regular** if Γ is both edge-regular and co-edge-regular.

To specify the value of k, we say that a graph is k-regular or regular with **valency** k. Similarly, we say that a graph is λ-edge-regular or μ-co-edge-regular, or that it is edge-regular, co-edge-regular or strongly regular with **parameters** (n, k, λ), (n, k, μ) or (n, k, λ, μ) respectively.

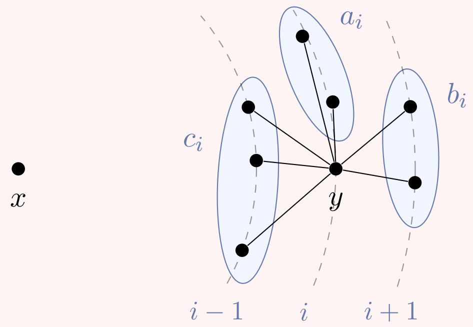
We require edge-regular graphs to be edgeless, since otherwise the edgeless graphs would fulfil the definition of edge-regularity for every value of λ and we could not speak of "the" parameters of an edge-regular graph anymore. Similarly, we consider complete graphs not to be co-edge-regular.

If a graph Γ is k-regular, then its complement $\overline{\Gamma}$ is also regular, with valency $n - k - 1$. If Γ is λ-edge-regular, then $\overline{\Gamma}$ is $(n + \lambda - 2k)$-co-edge-regular. Similarly, if Γ is μ-co-edge-regular, its complement is $(n + \mu - 2k - 2)$-edge-regular. In particular, the complement of a strongly regular graph parameters (n, k, λ, μ) is also strongly regular, with parameters $(n, n - k - 1, n + \mu - 2k - 2, n + \lambda - 2k)$.

Definition 1.7

A connected graph with diameter d is **distance-regular** if there exist constants a_i, b_i and c_i for all $i \in \{0, 1, \ldots, d\}$ such that the following hold for every pair of vertices v and w at distance i.

- a_i equals the number of vertices adjacent to v and at distance i from w.
- b_i equals the number of vertices adjacent to v and at distance $i+1$ from w.
- c_i equals the number of vertices adjacent to v and at distance $i-1$ from w.



The sequence $(b_0, b_1, \ldots, b_{d-1};\ c_1, c_2, \ldots, c_d)$ is called the **intersection array** of the graph.

A distance-regular graph is always regular: every vertex has b_0 neighbours. Moreover, a distance-regular graph is a_1-edge-regular. Notice that $b_0 = a_i + b_i + c_i$ for all $i \in \{0, 1, \ldots, d\}$ by b_0-regularity, which allows us to write the a_i's in function of the b_i's and c_i's. We conclude that, in order to prove distance-regularity, it suffices to check only the numbers of the intersection array.

Theorem 1.8 ([11])

Distance-regular graphs with diameter 2 are exactly the connected strongly regular graphs.

Proof. Let Γ be a distance-regular graph with diameter 2. We already noted that Γ is b_0-regular and a_1-edge-regular. Nonadjacent vertices are at distance 2 from each other, so Γ is c_2-co-edge-regular.

Let Γ be a connected strongly regular with parameters (n, k, λ, μ). Since Γ is not complete (by definition), there exist vertices v and w with $v \not\sim w$. Consider a walk $(v = v_0, v_1, v_2, \ldots, w)$ from v to w (such a walk exists by the connectedness of Γ). Then v_0 and v_2 are nonadjacent but have at least one common neighbour, so $\mu \geq 1$. In particular, $\operatorname{diam}(\Gamma) = 2$. One checks that Γ is distance-regular with intersection array $(k, k-\lambda-1;\ 1, \mu)$. $\square$

The spectrum of a graph plays a crucial role in this thesis. We will study cospectral graphs in Chapter 2.

Definition 1.9

Let Γ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** of Γ is the matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, where

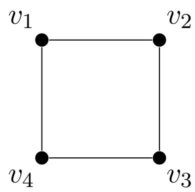
$$a_{ij} = \begin{cases} 0 & \text{if } v_i \not\sim v_j \\ 1 & \text{if } v_i \sim v_j. \end{cases}$$

The **spectrum** of Γ is the multiset of all eigenvalues of A.

We use the common convention of denoting the identity matrix by I and the all-one matrix by J. If we want to specify their dimension n, we also write I_n and J_n. Note that if A is the adjacency matrix of a graph, then $\overline{A} = J - A - I$ is the adjacency matrix of its complement.

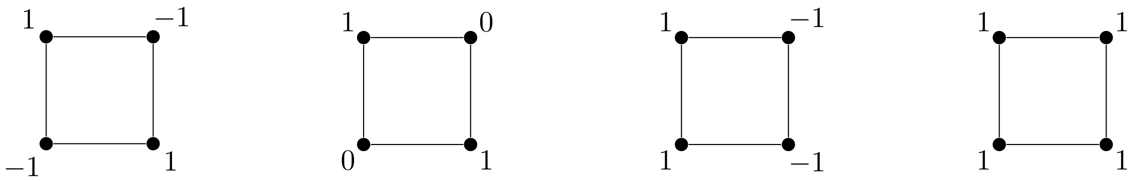
For an arbitrary vector $\vec{x} = (x_1, x_2, \ldots, x_n)^T$, the ith position of $A\vec{x}$ equals the sum of the x_j for which $v_i \sim v_j$. So if $\vec{x}$ is an eigenvector with eigenvalue λ and if we assign to each vertex v_i the value x_i, then the sum of the values of the neighbours of a vertex is equal to λ times the value of that vertex.

Example 1.10. Consider the cycle graph C_4 and its adjacency matrix A.



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of A is $\det(\lambda I - A) = \lambda^2(\lambda + 2)(\lambda - 2)$. So the spectrum of the graph is given by the multiset $\{-2, 0, 0, 2\}$. The basis of eigenvectors $(1, -1, 1, -1)^T$, $(1, 0, -1, 0)^T$, $(1, -1, -1, 1)^T$ and $(1, 1, 1, 1)^T$ can be represented as follows.



Notice how the values of the neighbours of a vertex indeed sum up to λ times the value of that vertex, where λ is the eigenvalue. This provides a second, more insightful method to find eigenvalues, without having to calculate the characteristic polynomial.

If we label the vertices of the graph in a different order, then the adjacency matrix changes (most of the time). So "the" adjacency matrix of a graph is actually not uniquely determined. However, it is determined up to conjugation with a permutation matrix. The following theorem implies that the spectrum does not depend on it. In particular, isomorphic graphs have the same spectrum.

Theorem 1.11 ([50])

Two adjacency matrices A and B have the same spectrum if and only if they are similar, i.e. if there exists an orthogonal matrix Q with $Q^T AQ = B$.

Since the adjacency matrix is real symmetric, the eigenvalues are real [38]. Moreover, their algebraic and geometric multiplicity are the same, which allows us to call it *the* multiplicity for short. The symmetry also informs us that eigenvectors with different eigenvalues are orthogonal.

Lemma 1.12 ([38])

A k-regular graph has eigenvalue k with multiplicity 1.

Proof. The eigenvalue k corresponds to the all-one eigenvector $(1, 1, \ldots, 1)^T$. Suppose that there is another eigenvector $\vec{x} = (x_1, x_2, \ldots, x_n)^T$ such that $A\vec{x} = k\vec{x}$. Let $j \in \{1, 2, \ldots, n\}$ be an index such that x_j is largest among all x_i's. Then

$$kx_j = (A\vec{x})_j = \sum_{v_i \sim v_j} x_i \leq kx_j$$

since $x_i \leq x_j$ for all $i \in \{1, 2, \ldots, n\}$. We have equality in every step, so $x_i = x_j$ must hold for all $i \in \{1, 2, \ldots, n\}$. We conclude that the eigenspace of k is spanned by the all-one vector. □

If a graph is k-regular, then the eigenvalue k is often called the *trivial eigenvalue*.

Lemma 1.13 ([38])

Let Γ be a k-regular graph with eigenvalues $k, \lambda_2, \ldots, \lambda_n$. Then $\overline{\Gamma}$ has eigenvalues $n - k - 1, -\lambda_2 - 1, \ldots, -\lambda_n - 1$.

Proof. Let $\overline{A}$ be the adjacency matrix of $\overline{\Gamma}$, i.e. $\overline{A} = J - A - I$. Let $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ be a basis of eigenvectors of A with eigenvalues $k, \lambda_2, \ldots, \lambda_n$ respectively, where $\vec{x}_1 = (1, 1, \ldots, 1)^T$. The all-one vector $\vec{x}_1$ is an eigenvector of $\overline{A}$ with eigenvalue $n - k - 1$ because $\overline{\Gamma}$ is regular with valency $n - k - 1$. For every $\vec{x}_i$ with $i \geq 2$, we have $\overline{A}\vec{x}_i = (J - A - I)\vec{x}_i = -A\vec{x}_i - \vec{x}_i = (-\lambda_i - 1)\vec{x}_i$, where we used that $J\vec{x}_i = 0$ since $\vec{x}_1$ and $\vec{x}_i$ are orthogonal. □

1.2 Finite projective spaces

The generalized Grassmann graphs in Chapter 3 are based on vector spaces over a finite field. It is often easier to reason in the associated projective space of a given vector space V than in V itself. In this section, we give some background on projective spaces. For more information on projective geometries in general, see [15].

Let $\mathbb{F}_q$ be the finite field of order q. Then q is a prime power, i.e. $q = p^h$ with p prime and $h > 0$.

Definition 1.14

Let V be a vector space. The **projective space $\mathbf{PG}(\boldsymbol{V})$** is the tuple $(D, \mathcal{I})$, where D is the set of all subspaces of V and $\mathcal{I} \subseteq D^2$ is the strict inclusion relation on D. Elements of D are again called **subspaces** and $\mathcal{I}$ is called the **incidence relation**. The **projective dimension** of a subspace in $\mathrm{PG}(V)$ is one less than its vector dimension in V.

If V is an n-dimensional vector space over the field $\mathbb{K}$, we denote $\mathrm{PG}(V) = \mathrm{PG}(n-1, \mathbb{K})$. If moreover $\mathbb{K} = \mathbb{F}_q$, we denote $\mathrm{PG}(V) = \mathrm{PG}(n-1, q)$.

Note that the subspaces of dimension 0, 1, 2, 3 and $n-1$ in a vector space V of dimension n are subspaces of projective dimension -1, 0, 1, 2 and $n-2$ in $\mathrm{PG}(V)$, respectively. We denote these subspaces by: the *empty set*, *points*, *lines*, *planes* and *hyperplanes*, respectively.

Formulas are often simpler using vector space dimensions, while most concepts are easier to picture projectively. We therefore adopt the following convention.

Dimensions are *vectorial* dimensions, unless stated otherwise.
Points, lines, etc. (also in figures) denote *projective* points and lines.

We also use the abbreviation “k-space” for a subspace of (vectorial) dimension k. In a similar fashion, we denote a set of size k as a “k-set”. Same for k-subsets and k-subspaces.

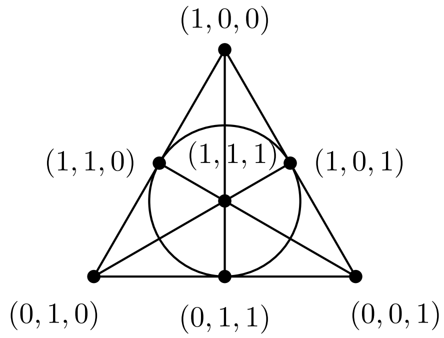
The equivalence of 1-spaces and points can be realized by homogeneous coordinates:

Definition 1.15

Fix a basis of $V \cong \mathbb{K}^n$ and let p be a point in $\mathrm{PG}(V)$, i.e. a 1-space in V. Let $(x_1, x_2, \ldots, x_n)^T$ be a nonzero vector in p. Then $(x_1, x_2, \ldots, x_n)$ is called a **homogeneous coordinate** of p.

A homogeneous coordinate is determined up to a factor $k \in \mathbb{K} \setminus \{0\}$. We will often define a point by its coordinate.

Example 1.16. $\mathrm{PG}(2, 2)$ is also known as the *Fano plane*. We can assign a coordinate to each point in the following way.



A useful identity is the so-called Grassmann formula. Though we defined dimensions to be vectorial, this formula also works projectively.

Lemma 1.17 (Grassmann formula, [8])

Let U and V be subspaces of a vector space. Then

$$\dim(U) + \dim(V) = \dim(U \cap V) + \dim(\langle U, V \rangle).$$

The Gaussian coefficient provides a q-analogue for the binomial coefficient. A q-analogue is a generalized expression of a statement with q instead of 1, that reduces to the original statement when we take the limit $q \to 1$. The value of q is typically a prime power [6].

Definition 1.18

The **Gaussian coefficient** or **q-binomial coefficient** is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}$$

for $0 \leq k \leq n$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q := 0$ otherwise.

For $k = 0$, the Gaussian coefficient equals 1, since both the numerator and denominator are empty products.

Recall that the number of k-sets in a given n-set is equal to $\binom{n}{k}$. A similar property holds for k-spaces in a given n-space. With "intersecting trivially", we mean that the intersection is the zero vector (similar to disjointness for sets).

Lemma 1.19 ([11])

Consider the vector space $\mathbb{F}_q^n$.

(i) The number of k-spaces in $\mathbb{F}_q^n$ equals $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

(ii) The number of k-spaces through a given m-space in $\mathbb{F}_q^n$ equals $\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q$.

(iii) The number of k-spaces that intersect a given m-space of $\mathbb{F}_q^n$ trivially equals $q^{mk}\begin{bmatrix} n-m \\ k \end{bmatrix}_q$.

Proof. (i) Suppose $1 \leq k \leq n$. If not, the statement is trivially true. Let a be the number of k-spaces in $\mathbb{F}_q^n$. Count the number of elements of

$$\left\{(\pi, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k) \mid \pi \text{ is a } k\text{-space of } \mathbb{F}_q^n \text{ with basis } \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}\right\}$$

in two ways (from left to right and vice versa). Then

$$a \cdot (q^k - 1)(q^k - q) \cdots (q^k - q^{k-1}) = (q^n - 1)(q^n - q) \cdots (q^n - q^{k-1}),$$

which implies $a = \begin{bmatrix} n \\ k \end{bmatrix}_q$.

(ii) Suppose $2 \leq m < k \leq n$. If not, the statement is trivially true. Let b be the number of k-spaces through a fixed m-space with basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ in $\mathbb{F}_q^n$. Count the number of elements of

$$\left\{(\pi, \vec{x}_{m+1}, \vec{x}_{m+2}, \ldots, \vec{x}_k) \mid \pi \text{ is a } k\text{-space of } \mathbb{F}_q^n \text{ with basis } \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}\right\}$$

in two ways (from left to right and vice versa). Then

$$b \cdot (q^k - q^m)(q^k - q^{m+1}) \cdots (q^k - q^{k-1}) = (q^n - q^m)(q^n - q^{m+1}) \cdots (q^n - q^{k-1}),$$

which implies $b = \begin{bmatrix} n-m \\ k-m \end{bmatrix}_q$.

(iii) Suppose $1 \leq k \leq n - m$. If not, the statement is trivially true. Let c be the number of k-spaces that intersect a fixed m-space of $\mathbb{F}_q^n$ trivially. Count the number of elements of

$$\left\{(\pi, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k) \mid \pi \text{ is a } k\text{-space of } \mathbb{F}_q^n \text{ with basis } \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}\right\}$$

in two ways (from left to right and vice versa). Then

$$c \cdot (q^k - 1)(q^k - q) \cdots (q^k - q^{k-1}) = (q^n - q^m)(q^n - q^{m+1}) \cdots (q^n - q^{m+k-1}),$$

which implies $c = q^{mk}\begin{bmatrix} n-m \\ k \end{bmatrix}_q$. □

In particular, the number of points in $\mathrm{PG}(n-1, q)$ is equal to $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1$.

An alternative proof of Lemma 1.19(ii) can be given by working in the so-called residual projective space of the m-space:

Definition 1.20

Let V be a finite-dimensional vector space and let W be a subspace of V. The **residual projective space $\mathrm{PG}_W(V)$** of W in $\mathrm{PG}(V)$ is the tuple $(D, \mathcal{I})$, where D is the set of all subspaces of V through W and $\mathcal{I} \subseteq D^2$ is the strict inclusion relation on D. The **residual dimension** of a subspace U in $\mathrm{PG}_W(V)$ is equal to $\dim(U) - \dim(W)$.

We call this structure a projective space, which is justified by the following theorem. By "isomorphic", we mean that there exists a bijection that preserves incidence and dimensions.

Theorem 1.21 ([43])

Let V be a finite-dimensional vector space and let W be a subspace of V. The residual projective space $\mathrm{PG}_W(V)$ is isomorphic to $\mathrm{PG}(W')$, where W' is a complement of W.

Proof (sketch). The correspondence is given by the projection map on W'. □

We end this section with a lemma that tells us something about the "sparsity" of finite projective spaces. Roughly speaking, these spaces are large enough to contain many trivially intersecting subspaces.

Lemma 1.22 ([9])

Let $k + m \leq n$. Given at most $q^{n-k-m+1}$ k-spaces in $\mathbb{F}_q^n$, we can always find an m-space that intersects them trivially.

Proof. Let $a \leq q^{n-k-m+1}$ be the number of given k-spaces in $\mathbb{F}_q^n$. We construct a suitable m-space by finding an ordered basis of vectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ for it. For the first vector, $\vec{x}_1$, there are at least

$$(q^n - 1) - a(q^k - 1)$$

choices: the total number of nonzero vectors, minus those that lie in one (and possibly more) of the a given k-spaces. There are at least

$$(q^n - 1) - (q - 1) - a(q^{k+1} - q)$$

choices for $\vec{x}_2$: the total number of nonzero vectors, minus those that span the same 1-space as $\vec{x}_1$, minus those that lie in the span of $\vec{x}_1$ and one of the given k-spaces, but are no multiple of $\vec{x}_1$ (because we eliminated those already).

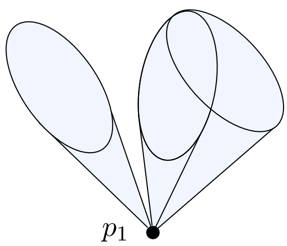


Figure 1.3. Forbidden area for the point $p_2 = \langle \vec{x}_2 \rangle$, where p_1 denotes $\langle \vec{x}_1 \rangle$.

Continuing in this way, we find a decreasing number of available vectors, ending with at least

$$(q^n - 1) - (q^{m-1} - 1) - a(q^{k+m-1} - q^{m-1})$$

choices for the last vector. Since $a \leq q^{n-k-m+1}$, this number is at least $q^n - q^{m-1} - q^n + q^{n-k} = q^{n-k} - q^{m-1}$, which is strictly positive since $k + m \leq n$. We conclude that such a space exists. ☐

1.3 Finite point-line geometries

A point-line geometry is a special type of incidence geometry with only two sorts of objects: points and lines (sometimes called blocks). The content of this section is based on [15]. It will play a large role in Chapter 5.

Definition 1.23

A **point-line geometry** is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, where $\mathcal{P}$ is a set of elements called **points**, $\mathcal{L}$ is a set of elements called **lines** and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$ is a so-called **incidence relation** such that for every $L \in \mathcal{L}$, there exist $p, q \in \mathcal{P}$, $p \neq q$, with $(p, L), (q, L) \in I$.

If $(p, L) \in \mathcal{I}$, we say that p and L are **incident**, p **lies on** L or L **goes through** p. Points are **collinear** if they lie on a common line. Lines are **concurrent** if they go through a common point.

We are mainly interested in *finite* point-line geometries, i.e. point-line geometries with a finite number of points and lines.

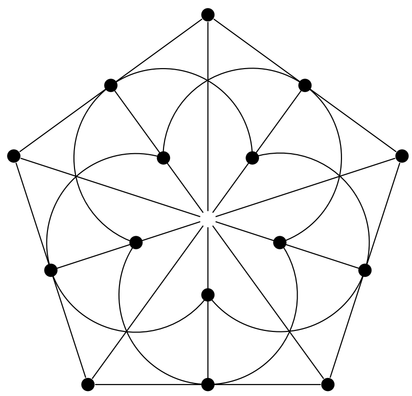
Many of the geometries that we will consider, satisfy the following property.

Definition 1.24

A **partial linear space** is a point-line geometry where two distinct points lie on at most one common line. It has **order (s, t)** if every line goes through exactly $s + 1$ points and every point lies on exactly $t + 1$ lines.

Example 1.25. Projective spaces are partial linear spaces (in the obvious way, where incidence is inclusion). Moreover, $\mathrm{PG}(n-1, q)$ has order $(q, \frac{q^{n-1}-1}{q-1})$ by Lemma 1.19.

Example 1.26. There are a ton of well-known examples of point-line geometries. The below geometry is also known as the *Doily*. As we will see in Example 3.2, it is related to a particular Kneser graph.



Lemma 1.27 ([15])

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a partial linear space of order (s, t). Then $|\mathcal{P}| \cdot (t+1) = |\mathcal{L}| \cdot (s+1)$.

Proof. Perform a double counting of incident point-line pairs. □

The definition of a partial linear space is very broad. For example, all graphs are partial linear spaces by interpreting vertices as points and edges as lines (and they have order $(1, k-1)$ if they are k-regular). Conversely, we can attach a graph to every finite point-line geometry through the following definition.

Definition 1.28

Let $\mathcal{S}$ be a finite point-line geometry.

- The **point graph** of $\mathcal{S}$ is the graph that has the points of $\mathcal{S}$ as its vertices and where two distinct points are adjacent if they are collinear. The **point matrix** of $\mathcal{S}$ is the adjacency matrix of the point graph of $\mathcal{S}$.
- The **line graph** of $\mathcal{S}$ is the graph that has the lines of $\mathcal{S}$ as its vertices and where two distinct lines are adjacent if they go through a common point. The **line matrix** of $\mathcal{S}$ is the adjacency matrix of the line graph of $\mathcal{S}$.
- The **incidence matrix** of $\mathcal{S}$ is the matrix $N = \left(\delta_{p,L}\right)_{p\in\mathcal{P}, L\in\mathcal{L}}$ indexed by the points and lines, where
$$\delta_{p,L} := \begin{cases} 0 & \text{if } (p, L) \notin \mathcal{I} \\ 1 & \text{if } (p, L) \in \mathcal{I}. \end{cases}$$

Note that the incidence matrix is in general not a square matrix.

Much like the adjacency matrix of a graph, the point matrix, line matrix and incidence matrix of a point-line geometry are determined up to a permutation of the points and lines. However, we will assume that their rows and columns are labelled in the same order.

We end this chapter with the following lemma.

Lemma 1.29 ([26])

Let $\mathcal{S}$ be a finite partial linear space of order (s, t) with point matrix P, line matrix L and incidence matrix N. Then $NN^T = P + (t+1)I$ and $N^T N = L + (s+1)I$.

Proof. Let p, q be two points. Then
$$\left(NN^T\right)_{p,q} = \sum_{M\in\mathcal{L}} \delta_{p,M}\delta_{q,M} = \begin{cases} t+1 & \text{if } p = q \\ (P)_{p,q} & \text{if } p \neq q \end{cases}$$
since p and q are adjacent in the point graph if and only if they lie on exactly one common line M. We conclude that $NN^T = P + (t+1)I$. The proof of $N^T N = L + (s+1)I$ is almost identical. □

Chapter 2

Spectral graph theory

The purpose of this chapter is to give some background on spectral graph theory. As the title suggests, we will be looking at the spectrum of a graph (see Definition 1.9). We continue with a description of some useful methods to construct graphs with the same spectrum. Most information on graph spectra can also be found in [12].

2.1 Cospectral graphs

Our interest goes to pairs of graphs that have the same spectrum. They play a large role in this thesis.

Definition 2.1

Graphs are **cospectral** if they have the same spectrum. **Cospectral mates** are nonisomorphic cospectral graphs. A graph is **determined by its spectrum** (DS) if it has no cospectral mate.

In other words, a graph is determined by its spectrum if every graph cospectral with it, is isomorphic to it. A graph that has a cospectral mate, is not determined by its spectrum (NDS). The property that indicates whether a graph is DS or NDS, will be denoted as the "cospectrality" of that graph.

Quite some properties of a graph (such as the number of closed walks of fixed length, or strongly regularity) can be distilled from its spectrum. In that way, the spectrum acts like a footprint of the graph. It is natural to ask whether one can reconstruct a graph from its spectrum. However, not all graphs are determined by their spectrum. The smallest example of cospectral mates is the so-called *Saltire pair*, named after the Scottish flag, Saltire, since the two graphs superposed form its shape. It was first reported in 1957 [21].

Figure 2.1. The Saltire pair.

Both graphs have spectrum $\{-2, 0, 0, 0, 2\}$. Notice how the first graph, $C_4 \cup K_1$, has a basis of eigenvectors that are the eigenvectors of C_4 (see Example 1.10) extended by a 0 on the middle vertex, together with the eigenvector that is 1 on the middle vertex and 0 on the others. A basis of eigenvectors for the second graph can be retrieved in a similar fashion.

It has been proved that there exist an infinite amount of cospectral mates. For example, almost all trees are cospectral [60]. We will prove an asymptotic lower bound on the number of cospectral graphs in Chapter 11.

Still, most "common" graphs seem to be DS. We give some examples in the list below.

Theorem 2.2 ([12, Section 14.4])

The following graphs are determined by their spectrum (DS).

(i) The path graph P_n.

(ii) The cycle graph C_n.

(iii) The disjoint union of complete graphs, $K_{m_1} \cup \cdots \cup K_{m_k}$.

(iv) The complete bipartite graph $K_{m,m}$.

(v) The complement of a regular graph that is DS.

(vi) The disjoint union of several copies of a strongly regular graph.

These observations make the following seem plausible.

Conjecture 2.3 ([26])

Almost all graphs are determined by their spectrum.

In other words, though there are an infinite number of cospectral mates, their proportion would tend to zero. The conjecture was given numerical support in 2009 by Brouwer and Spence [14]. They calculated the spectrum of all graphs with up to 12 vertices and saw that the fraction of graphs that are determined by their spectrum increases between 10 and 12 vertices. Moreover, there is also an increasing amount of theoretical evidence that the conjecture might be true.

2.2 Spectra of other matrices

In some contexts, the spectrum of a graph might denote the spectrum of an other matrix associated with it. Though the adjacency matrix is the most common choice, it can also be interesting to look at the spectrum of the following matrices.

Definition 2.4

Let Γ be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. Let D be the diagonal matrix containing the degrees of Γ, i.e. $D := \operatorname{diag}\big(\deg(v_1), \deg(v_2), \ldots, \deg(v_n)\big)$. The **Laplacian matrix** of Γ is the matrix $L := D - A$. The **signless Laplacian matrix** of Γ is the matrix $|L| := D + A$. The **Seidel matrix** of Γ is the matrix $S := \overline{A} - A$, where $\overline{A}$ is the adjacency matrix of $\overline{\Gamma}$.

We can also investigate cospectrality with respect to these matrices. Mind that, when we talk about just "the spectrum", we always mean the spectrum of the adjacency matrix, unless specified otherwise.

2.3 Techniques for finding cospectral graphs

In this thesis, we focus on constructing cospectral graphs, and less on proving that the spectrum determines the graph, because the latter is often difficult. Constructing cospectral nonisomorphic graphs can give us a better understanding of the limitations of Conjecture 2.3.

Note that, though the below methods provide cospectral graphs, most of them do not guarantee nonisomorphism. When we will construct cospectral mates in order to show that a graph is NDS, proving nonisomorphism is mostly the hardest part of the problem.

The idea of switching was first introduced by Seidel in 1966 [52].

2.3.1 Seidel switching

Searching for cospectral mates with respect to the Seidel matrix $S = \overline{A} - A = J - I - 2A$ is made easy by the operation of Seidel switching.

Definition 2.5 ([52])

Let Γ be a graph and $C \subseteq V(\Gamma)$. For every $u \in C$ and every $v \notin C$, reverse the adjacency between u and v. This operation is called **Seidel switching**. We say that the resulting graph is obtained from Γ by Seidel switching with respect to C.

Two graphs are **Seidel equivalent** if they can be obtained from one another by Seidel switching.

In other words, Seidel switching with respect to C returns the graph (V, E) with

$$\begin{aligned} V &= V(\Gamma) \\ E &= E(\Gamma) \ \cup \ \big\{\{u, v\} \mid u \in C \text{ and } v \notin C\big\} \ \setminus \ \big\{\{u, v\} \in E(\Gamma) \mid u \in C \text{ and } v \notin C\big\}\,. \end{aligned}$$

Example 2.6. The following graphs are Seidel equivalent by switching with respect to the set C.

It is always possible to find a Seidel equivalent graph with an isolated vertex: choose any vertex and let C be the set of its neighbours. The previous example illustrates this for the top vertex.

Note that Seidel equivalence is indeed an equivalence relation. Reflexivity follows from taking $C = \emptyset$ or $C = V(\Gamma)$. Symmetry is due to the fact that the operation is involutive: switching two times with respect to the same set results again in the original graph. Switching first with respect to C_1 and then with respect to C_2 is the same as switching with respect to their symmetric difference $C_1 \triangle C_2$. So transitivity holds as well.

Example 2.7. The switching equivalence class of the complete graph K_n consists of all unions $K_{n_1} \cup K_{n_2}$ with $n_1 + n_2 = n$.

Seidel equivalent graphs share their Seidel spectrum, as shown by the following theorem.

Theorem 2.8 ([52])

Seidel switching leaves the Seidel spectrum invariant.

Proof. Label the vertices of the graph in such a way that its adjacency matrix and Seidel matrix have block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \overline{A} - A = \begin{pmatrix} \overline{A_{11}} - A_{11} & \overline{A_{12}} - A_{12} \\ \overline{A_{21}} - A_{21} & \overline{A_{22}} - A_{22} \end{pmatrix},$$

where A_{11} and S_{11} correspond to the vertices of the switching set C. The graph obtained by switching with respect to C then has adjacency matrix and Seidel matrix

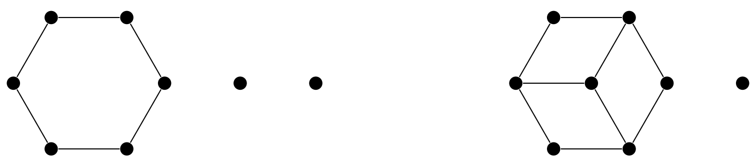
$$A' = \begin{pmatrix} A_{11} & \overline{A_{12}} \\ \overline{A_{21}} & A_{22} \end{pmatrix} \quad \text{and} \quad S' = \overline{A'} - A' = \begin{pmatrix} \overline{A_{11}} - A_{11} & A_{12} - \overline{A_{12}} \\ A_{21} - \overline{A_{21}} & \overline{A_{22}} - A_{22} \end{pmatrix} = \begin{pmatrix} S_{11} & -S_{12} \\ -S_{21} & S_{22} \end{pmatrix}.$$

Since

$$\begin{pmatrix} S_{11} & -S_{12} \\ -S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}^T \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$$

the Seidel matrices S and S' are similar and therefore cospectral. □

The other direction is in general not true: graphs with the same Seidel spectrum are not necessarily Seidel equivalent. Counterexamples start occurring for graphs with eight vertices. One such pair is given by the following two graphs. They both have Seidel spectrum $\{-3, -3, 1-2\sqrt{3}, -1, 1, 1, 3, 1+2\sqrt{3}\}$ but are not Seidel equivalent.



Note that when a graph has at least two vertices, there is always a Seidel switching that changes the number of edges in the graph (in order to see this, take any two vertices u and v and consider the switching sets $\{u\}$,$\{v\}$ and $\{u, v\}$), leading to at least one nonisomorphic graph with the same Seidel spectrum. So in the case of the Seidel matrix, having a cospectral mate is a trivial property and in particular, Conjecture 2.3 is false for the Seidel spectrum. Instead, we could change it to "Which Seidel switching classes are determined by their spectrum?" where the classes are the equivalence classes of the Seidel equivalence relation. This is again an open problem.

In some cases, Seidel switching can provide cospectral graphs for the adjacency matrix as well. For example, if Γ is a k-regular graph such that the graph Γ' obtained by switching is also k-regular, then Γ and Γ' are cospectral. This specific case of Seidel switching is actually an example of GM-switching, and will therefore be proved in the next section.

2.3.2 GM-switching

Godsil and McKay presented several techniques for finding pairs of nonisomorphic cospectral graphs in [37]. The most famous one is probably Godsil-McKay switching, or GM-switching for short.

Definition 2.9 ([37])

Let Γ be a graph and let $\{C_1, C_2, \ldots C_k, D\}$ be a partition of $V(\Gamma)$ such that the following hold for all $i, j \in \{1, 2, \ldots, k\}$:

(i) Any two vertices of C_i have the same number of neighbours in C_j.

(ii) Every vertex in D has exactly 0, $\frac{1}{2}|C_i|$ or $|C_i|$ neighbours in C_i.

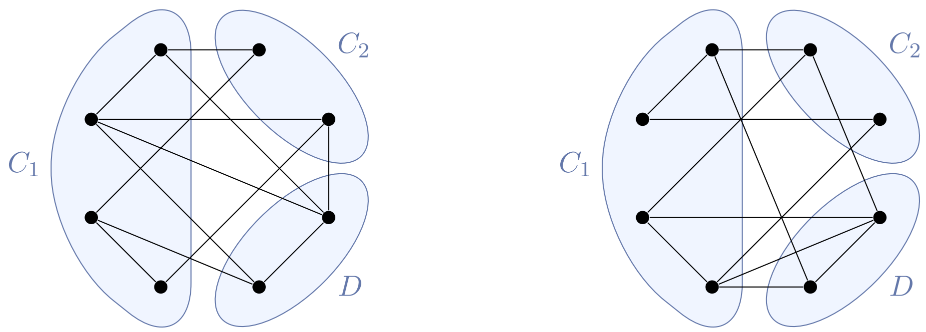
For all $i \in \{1, 2, \ldots, k\}$, every $u \in C_i$ and every $v \in D$ that has exactly $\frac{1}{2}|C_i|$ neighbours in C_i, reverse the adjacency between u and v. This operation is called **GM-switching**. We say that the resulting graph is obtained from Γ by GM-switching with respect to $C_1, C_2, \ldots C_k$.

Formally, the graph obtained from Γ by GM-switching with respect to $C_1, C_2, \ldots C_k$ is the graph (V, E) with

$$
\begin{aligned}
V &= V(\Gamma) \\
E &= E(\Gamma) \cup \left\{ \{u, v\} \mid i \in \{1, 2, \ldots, k\}, u \in C_i, v \in D \text{ and } v \text{ has } \frac{1}{2}|C_i| \text{ neighbours in } C_i \right\} \\
&\quad \setminus \left\{ \{u, v\} \in E(\Gamma) \mid i \in \{1, 2, \ldots, k\}, u \in C_i, v \in D \text{ and } v \text{ has } \frac{1}{2}|C_i| \text{ neighbours in } C_i \right\}.
\end{aligned}
$$

In other words, for every vertex $v \in D$ that is adjacent to exactly half of the vertices of C_i, delete these edges and instead add the other ones.

Example 2.10. Consider the graph on the left and the graph obtained from it by GM-switching with respect to C_1 and C_2.



Notice how the resulting graph is not isomorphic since on the left, all three triangles have a common edge, while on the right they do not. Also note how an isomorphic graph to the one on the right can be obtained by switching the left one with respect to only C_1, because C_2 merely interchanges the roles of its two vertices. More generally, sets C_i of size 2 in the partition do not play any significant role since they just interchange the roles of their two vertices, so they can as well be absorbed by D.

The power of GM-switching is the following.

Theorem 2.11 ([37])

GM-switching leaves the spectrum invariant.

Proof. Label the vertices of the graph in such a way that its adjacency matrix has a block form that is consistent with the partition of its vertex set into the switching sets $C_1, C_2, \dots C_k$ and their complement D. The adjacency matrices of the original graph and the graph obtained by switching then have the form

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1,k} & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{k,1} & \cdots & A_{k,k} & B_k \\ B_1^T & \cdots & B_k^T & M \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} A_{11} & \cdots & A_{1,k} & B'_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{k,1} & \cdots & A_{k,k} & B'_k \\ {B'_1}^T & \cdots & {B'_k}^T & M \end{pmatrix}$$

respectively, where B'_i denotes the matrix obtained from B_i by replacing each column that has $\frac{1}{2}|C_i|$ ones by its complement, i.e. $B'_i = Q_i B_i$, where $Q_i = \frac{2}{|C_i|}J - I$ is an involutive square matrix of size $|C_i|$. Furthermore, since A_{ij} has constant row and column sums, $Q_i A_{ij} Q_j = A_{ij}$. Thus,

$$\begin{pmatrix} A_{11} & \cdots & A_{1,k} & B'_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{k,1} & \cdots & A_{k,k} & B'_k \\ {B'_1}^T & \cdots & {B'_k}^T & M \end{pmatrix} = \begin{pmatrix} Q_1 & \cdots & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & Q_k & O \\ O & \cdots & O & I \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{1,k} & B_1 \\ \vdots & \ddots & \vdots & \vdots \\ A_{k,1} & \cdots & A_{k,k} & B_k \\ B_1^T & \cdots & B_k^T & M \end{pmatrix} \begin{pmatrix} Q_1 & \cdots & O & O \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & Q_k & O \\ O & \cdots & O & I \end{pmatrix}$$

which means that the adjacency matrices A and A' are similar and therefore cospectral. □

Though GM-switching was invented for constructing graphs that are cospectral with respect to the adjacency matrix, the above proof also works for other matrices. The same argument holds for the Laplacian and signless Laplacian matrix, as well as for the adjacency matrix of the complement [26]. The latter is not very surprising, since the definition of GM-switching is self-complementary. We conclude that if two graphs are cospectral by GM-switching, then so are their complements.

In this work, we focus on the following simplified version of GM-switching. This case is probably the most interesting because it is quite simple, and yet it produces many cospectral graphs, as we will see in Chapter 11.

Definition 2.12 ([37])

Let Γ be a graph and $C \subseteq V(\Gamma)$ such that the following hold:

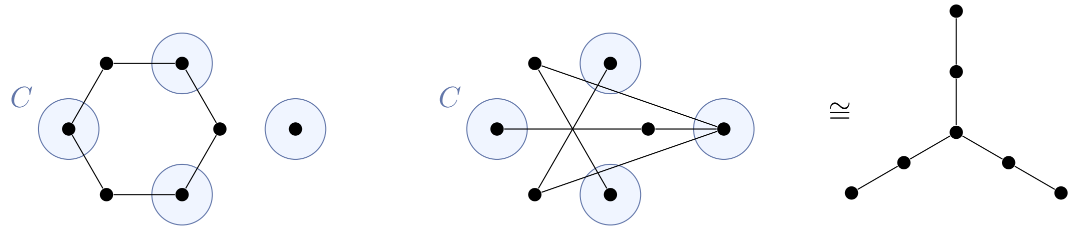
(i) The induced subgraph on C is regular.

(ii) Every vertex outside C has exactly 0, $\frac{1}{2}|C|$ or $|C|$ neighbours in C.

For every $u \in C$ and every $v \notin C$ that has exactly $\frac{1}{2}|C|$ neighbours in C, reverse the adjacency between u and v. We say that the resulting graph is obtained from Γ by **GM-switching** with respect to C.

We will always require $|C|$ to be even, since otherwise the graph is left untouched. GM-switching with respect to a set of size 2 is also trivial, because it merely interchanges the roles of the two vertices in the switching set. Switching sets of size 4 are far more interesting. There is even evidence that this is the most productive size to construct cospectral mates.

Example 2.13. The smallest cospectral mates that can be obtained by GM-switching have order 7. One of these pairs is given below. Both graphs have spectrum $\{-2, -1, -1, 0, 1, 1, 2\}$ but are not isomorphic.



Unlike Seidel switching, GM-switching is not transitive: applying the technique two times (with respect to different sets) does not always reduce to applying it once. A counterexample can be found in [2, Example 4]. Therefore, while searching for *multiple* cospectral mates, it can be useful to try and switch multiple times with respect to different sets. On the other hand, if one just wants to find *any* cospectral mate, one does not have to switch more than once.

2.3.3 WQH-switching

Another switching technique was provided by Wang, Qui and Hu in 2019 [63]. They call it generalized Godsil-McKay switching. We will refer to it as WQH-switching.

Definition 2.14 ([63])

Let Γ be a graph and let C_1, C_2 be disjoint subsets of $V(\Gamma)$ such that the following hold:

(i) $|C_1| = |C_2|$

(ii) There exists a constant c such that for all $i, j \in \{1, 2\}$, $i \neq j$ and every vertex of C_i, the number of neighbours in C_i minus the number of neighbours in C_j is equal to c .

(iii) Every vertex outside $C_1 \cup C_2$ has either:

   (a) $|C_1|$ neighbours in C_1 and 0 neighbours in C_2,
   (b) 0 neighbours in C_1 and $|C_2|$ neighbours in C_2,
   (c) the same number of neighbours in C_1 as in C_2.

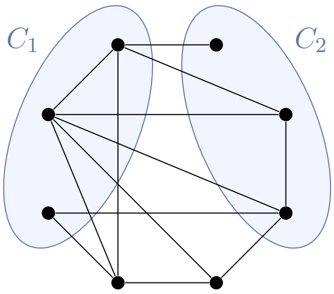
For every $u \in C_1 \cup C_2$ and every $v \notin C_1 \cup C_2$ for which (a) or (b) holds, reverse the adjacency between u and v. This operation is called **WQH-switching**. We say that the resulting graph is obtained from Γ by WQH-switching with respect to (C_1, C_2).

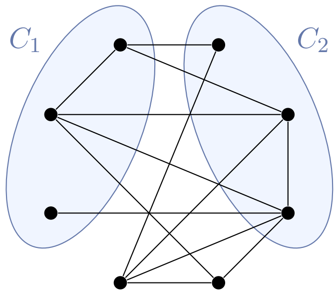
Although WQH-switching happens with two sets, we will often say that $C_1 \cup C_2$ is *the* switching set, and that we are switching with respect to a set of size $|C_1 \cup C_2| = 2|C_1|$.

Note that if if $|C_1| = |C_2| = 1$, WQH-switching with respect to (C_1, C_2) is trivial, since it just interchanges the roles of C_1 and C_2 (similar to GM-switching with respect to a set of size 2). If

$|C_1| = |C_2| = 2$, then WQH-switching with respect to (C_1, C_2) is the same as GM-switching with respect to a 4-set. Indeed, any partition of a GM-switching set of size 4 into two sets of size 2 suffices the conditions of Definition 2.14. On the other hand, $C_1 \cup C_2$ is always regular by Definition 2.14(ii) and WQH-switching with respect to C_1 and C_2 is the same as GM-switching with respect to $C_1 \cup C_2$ and then swapping the two vertices in C_1 and also those of C_2. For larger switching set sizes, WQH-switching is no longer the same as GM-switching. For example, if $C \cong C_5 \cup C_3$ in Definition 2.12, we cannot divide C into two switching sets that meet the conditions of WQH-switching. Conversely, $K_4 \cup K_2$ can be partitioned into two WQH-switching sets, but it is not regular.

Example 2.15. Consider the following two graphs. They are obtained from one another by WQH-switching with respect to (C_1, C_2).





Note that these graphs are not isomorphic. A possible explanation could be that the degree of the vertex that is adjacent to the unique vertex of degree one, is different. The below theorem implies that they are cospectral mates.

Theorem 2.16 ([63])

WQH-switching leaves the spectrum invariant.

Proof. Let $m = |C_1| = |C_2|$. Label the vertices of the graph in such a way that its adjacency matrix and the adjacency matrix of the graph obtained by switching with respect to (C_1, C_2), have block form

$$A = \begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ B_1^T & B_2^T & M \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} A_{11} & A_{12} & B_1' \\ A_{21} & A_{22} & B_2' \\ {B_1'}^T & {B_2'}^T & M \end{pmatrix},$$

where A_{11} corresponds to the vertices of C_1 and A_{22} to those of C_2. Then B_1' is the matrix obtained from B_1 by replacing an all-one column by an all-zero column if the corresponding column of B_2 is all-zero, by replacing an all-zero column by an all-one column if the corresponding column of B_2 is all-one and retaining the original column if it has equally many ones as the corresponding column of B_2. In other words, $B_1' = (I - \frac{1}{m}J)B_1 + \frac{1}{m}JB_2$. Similarly, $B_2' = \frac{1}{m}JB_1 + (I - \frac{1}{m}J)B_2$. Furthermore, because of property (ii), $(A_{11} - A_{12})J = (A_{22} - A_{21})J = J(A_{11} - A_{21}) = J(A_{22} - A_{12}) = cJ$. A straight-forward calculation implies that

$$\begin{pmatrix} A_{11} & A_{12} & B_1' \\ A_{21} & A_{22} & B_2' \\ {B_1'}^T & {B_2'}^T & M \end{pmatrix} = \begin{pmatrix} I - \frac{1}{m}J & \frac{1}{m}J & O \\ \frac{1}{m}J & I - \frac{1}{m}J & O \\ O & O & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ B_1^T & B_2^T & M \end{pmatrix} \begin{pmatrix} I - \frac{1}{m}J & \frac{1}{m}J & O \\ \frac{1}{m}J & I - \frac{1}{m}J & O \\ O & O & I \end{pmatrix}$$

so the adjacency matrices A and A' are similar and therefore cospectral. □

WQH-switching is a very recent technique that is gaining more attention nowadays. For example, it is used in [47] to construct a large number of graphs that are cospectral with line graphs of certain partial linear spaces.

2.3.4 Point-line geometries

An interesting class of cospectral graphs arises from point-line geometries. Consider a finite partial linear space with an order and equally many points and lines. Then its point graph and line graph are cospectral, and in many cases, nonisomorphic:

Lemma 2.17 ([38])

Let A and B be two (not necessarily square) matrices such that AB and BA are defined. Then AB and BA have the same nonzero eigenvalues with the same algebraic multiplicities.

Proof. Let $x \neq 0$ be a variable. Define $X := \begin{pmatrix} I & x^{-1}A \\ B & I \end{pmatrix}$ and $Y := \begin{pmatrix} I & O \\ -B & I \end{pmatrix}$. Then $\det(I - x^{-1}AB) = \det(XY) = \det(YX) = \det(I - x^{-1}BA)$. So the characteristic polynomial of AB and BA is the same up to a power of x. □

Theorem 2.18 ([12])

Let $\mathcal{S}$ be a finite partial linear space of order (s, t) with $s = t$. Then its point graph and line graph are cospectral.

Proof. We know from Lemma 1.29 that the point graph and line graph of $\mathcal{S}$ are given by $NN^T - (t+1)I$ and $N^TN - (s+1)I$, where N denotes the incidence matrix of $\mathcal{S}$. If $s = t$, then the point graph and line graph have the same nonzero eigenvalues with the same multiplicities, thanks to Lemma 2.17 with $A = N$ and $B = N^T$. But these graphs have the same order, since $|\mathcal{P}| \cdot (t+1) = |\mathcal{L}| \cdot (s+1)$ by Lemma 1.27. So the zero eigenvalues have the same multiplicity as well. □

Theorem 2.19 ([27])

Let $\mathcal{S}$ and $\mathcal{S}'$ be two finite partial linear spaces of the same order (s, t) and with the same point graph. Then their line graphs are cospectral.

Proof. Let P, L and N denote the point matrix, line matrix and incidence matrix of $\mathcal{S}$ respectively, and let P', L' and N' be those of $\mathcal{S}'$ (indexed in the same way). The point graphs of $\mathcal{S}$ and $\mathcal{S}'$ are the same, so $P = P'$ and $\mathcal{S}$ and $\mathcal{S}'$ have the same number of points and lines (the latter follows from Lemma 1.27). By Lemma 1.29, we have $NN^T = P + (t+1)I = N'N'^T$, $N^TN = L + (s+1)I$ and $N'^TN' = L' + (s+1)I$.

We now prove that L and L' have the same spectrum. Consider an arbitrary number $\lambda \neq -(s+1)$. Since $N^TN = L+(s+1)I$, λ is an eigenvalue of L with multiplicity m if and only if $\lambda+(s+1) \neq 0$ is an eigenvalue of N^TN with multiplicity m. By Lemma 2.17, this is equivalent with $\lambda+(s+1)$ being an eigenvalue of $NN^T = N'N'^T$ with multiplicity m, which holds if and only if λ is an eigenvalue of

L' with multiplicity m, because $N'^T N' = L' + (s+1)I$. So L and L' have the same spectrum, except maybe for the multiplicity of $-(s+1)$. But both matrices have the same size, so this multiplicity must be the same as well. □

Note that there are many more techniques for constructing cospectral graphs than the previous four methods, see e.g. [2, 30].

2.4 Spectral characterizations

We conclude this chapter with some useful spectral properties of regular graphs.

Theorem 2.20

Regular graphs are cospectral if and only if their complements are cospectral.

Proof. This follows directly from Lemma 1.13. □

Theorem 2.21 ([25])

Distance-regular graphs are cospectral if and only if they have the same intersection array.

In particular, strongly regular graphs are cospectral if and only if they have the same parameters. But there is more. We have the following beautiful characterization of strongly regular graphs in terms of their eigenvalues.

Theorem 2.22 ([38])

A graph is strongly regular if and only if it has exactly three eigenvalues. Moreover, its parameters are (n, k, λ, μ) if and only if its eigenvalues are equal to k and the two roots of $x^2 + (\mu - \lambda)x + \mu - k$.

As a direct corollary of this last theorem, we can use the techniques for constructing cospectral graphs in order to find new strongly regular graphs with the same parameters as existing ones. Since the latter is an important problem in graph theory, this strategy has been applied many times in the literature, see e.g. [1, 3, 13, 47, 48, 49].

Chapter 3

Generalized Johnson graphs and generalized Grassmann graphs

In this chapter, we introduce the graphs of our interest: the generalized Johnson and Grassmann graphs. These include the Kneser graphs, Johnson graphs and their q-analogues. In the following, n and k will denote natural numbers with $k \leq n$.

3.1 Generalized Johnson graphs

We begin by defining the "ordinary" Kneser graphs $K(n, k)$ and Johnson graphs $J(n, k)$, and work our way up to the larger group of generalized Johnson graphs.

3.1.1 Kneser graphs

Definition 3.1

The **Kneser graph $K(n, k)$** has as vertices the k-subsets of $\{1, 2, \dots, n\}$, where two vertices are adjacent if the corresponding k-subsets are disjoint.

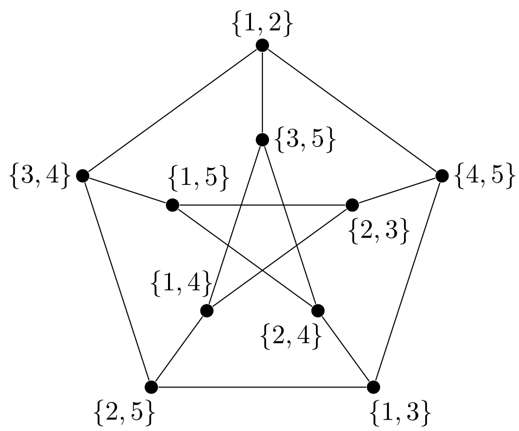


Figure 3.1. The Petersen graph $K(5, 2)$.

Kneser graphs were introduced in 1978 by Lovász in order to prove the Kneser conjecture [53] that was first stated by Kneser [51]. They have been studied very well for a long time. Other notations that can be found in the literature, are $K_{n:k}$, $KG_{n,k}$ and K_k^n [38, 53, 62].

$K(n,1)$ is the complete graph K_n, since distinct singletons are disjoint. If $k > n/2$, then any two k-sets have a nonempty intersection, in which case $K(n,k)$ is the edgeless graph $\overline{K_n}$. If $k = n/2$, then every k-set has a unique neighbour: its complement in $\{1,2,\ldots,n\}$. In order to eliminate these trivial cases, we often make the assumption that $2 \leq k < n/2$. With this in mind, the smallest nontrivial example of a Kneser graph is $K(5,2)$, the celebrated *Petersen graph* (Figure 3.1).

Example 3.2. The Doily (see Example 1.26) has point graph $K(6,2)$: we can assign to each point a pair such that three collinear points partition the set $\{1,2,3,4,5,6\}$.

$K(2n-1,n-1)$ is also called the *Odd graph* O_n. Odd graphs are determined by their spectrum, as we shall see in Result 4.3.

Lemma 3.3

Let $k \leq n/2$. The Kneser graph $K(n,k)$ is edge-regular with parameters $\left(\binom{n}{k}, \binom{n-k}{k}, \binom{n-2k}{k}\right)$.

Proof. The number of vertices of $K(n,k)$ is equal to the number of k-subsets of $\{1,2,\ldots,n\}$, i.e. $\binom{n}{k}$. The neighbours of a vertex v are the k-sets that are disjoint with it, being exactly those k-subsets of the $(n-k)$-set $\{1,2,\ldots,n\} \setminus v$. That makes $K(n,k)$ regular with valency $\binom{n-k}{k}$. Similarly, the common neighbours of two vertices v and w are the k-subsets of $\{1,2,\ldots,n\} \setminus (v \cup w)$, a set of size $n-2k$ if v and w are disjoint (adjacent). □

$K(n,k)$ is strongly regular if $k = 2$, but in general, it is not co-edge-regular. Kneser graphs are also highly symmetric, in the sense that their automorphism group contains the symmetric group on n elements. Moreover, the combinatorial definition of Kneser graphs makes them interesting, because they allow to translate combinatorial problems into graph theoretical problems. For example, the following theorem plays an important role in combinatorics.

Theorem 3.4 (Erdős-Ko-Rado theorem [33])

The independence number of the Kneser graph $K(n,k)$ is equal to $\alpha\left(K(n,k)\right) = \binom{n-1}{k-1}$.

The Erdős-Ko-Rado theorem can be proved by using Hoffman's ratio bound, a bound on the size of a coclique in terms of the size, valency and smallest eigenvalue of the graph [12].

3.1.2 Johnson graphs

Definition 3.5

The **Johnson graph $J(n,k)$** has as vertices the k-subsets of $\{1,2,\ldots,n\}$, where two vertices are adjacent if the corresponding k-subsets have $k-1$ elements in common.

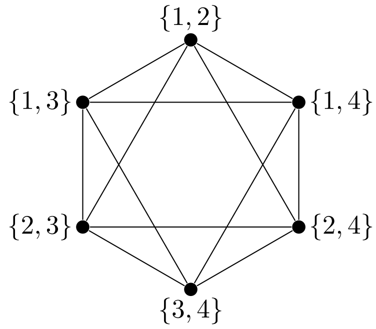


Figure 3.2. The octahedral graph $J(4, 2)$.

Example 3.6. $J(n, 2)$ is also called the *triangular graph* T_n. It is strongly regular with parameters $\left(\binom{n}{2}, 2(n-2), n-2, 4\right)$. The smallest nontrivial Johnson graph is the triangular graph T_4, which is also known as the *octahedral graph* (Figure 3.2). Note that $J(n, 2)$ and $K(n, 2)$ are complementary graphs. Indeed, any two distinct 2-sets meet in either one or no element. In particular, the Petersen graph is the complement of the triangular graph T_5.

Theorem 3.7 ([11, Theorem 9.1.2])

The Johnson graph $J(n, k)$ is distance-regular with diameter $d = \min(k, n-k)$ and intersection numbers

$$\begin{aligned} a_i &= i(n-2i), \\ b_i &= (k-i)(n-k-i), \\ c_i &= i^2. \end{aligned}$$

Moreover, $\mathrm{d}(v, w) = i \iff |v \cap w| = k - i$.

Proof. We first prove the expression for the distance. It suffices to show that $\mathrm{d}(v, w) \leq i \iff |v \cap w| \geq k - i$ (consider the same inequality for $i - 1$). We do this by induction on i. It is trivially true for $i = 0$ and $i = 1$. Now suppose it holds for i. Then it must also hold for $i + 1$ by the following argument.

If $\mathrm{d}(v, w) \leq i+1$, then there is a k-set u with $\mathrm{d}(u, v) \leq i$ and $u \sim w$. In other words, $|u \cap v| \geq k - i$ and $|u \cap w| = k - 1$. By the inclusion-exclusion principle, $k = |u| \geq |u \cap v| + |u \cap w| - |u \cap v \cap w|$, so $|v \cap w| \geq |u \cap v \cap w| \geq (k-i) + (k-1) - k = k - (i+1)$. Conversely, if $|v \cap w| \geq k - (i+1)$, then we choose an element $p \in v \setminus w$ and an element $q \in w \setminus v$. The k-set $u := w \setminus \{q\} \cup \{p\}$ intersects v in at least a $(k-i)$-set and w in a $k-1$-set, so $\mathrm{d}(v, w) \leq \mathrm{d}(u, v) + \mathrm{d}(u, w) \leq i + 1$.

The intersection of two k-sets is at least $\max(0, 2k - n)$, so the diameter equals $\min(k, n - k)$.

In order to determine the b_i's, consider two k-sets v and w at distance i and let u be a k-set such that $\mathrm{d}(u, v) = i + 1$ and $u \sim w$. Because of the latter condition, there is a unique element $p \in w \setminus u$, but the former implies that $p \in v \cap w$. So there are $|v \cap w| = k - i$ possibilities for p. On the other hand, there are $n - |v \cup w| = n - k - i$ choices left for the unique element of $u \setminus w$.

For the c_i's, consider two k-sets v and w at distance i and let u be a k-set such that $\mathrm{d}(u, v) = i-1$ and $u \sim w$. Similarly to the previous reasoning, there is a unique element $p \in w \setminus u$. Since $|u \cap v| = k-i+1$ and $|u \setminus w| = 1$, we see that necessarily $p \notin v$. So there are $k - (k - i) = i$ choices for p and $k - (k - i) = i$ more choices for the unique element of $u \setminus w$, which makes a total of $c_i = i^2$ possible such k-sets.

The formula for the a_i's follows by applying $b_0 = a_i + b_i + c_i$. □

The previous theorem provides an alternative way to define the Kneser graph $K(n,k)$, that is used in some other works, like [12]. That is, $K(n,k)$ is the maximal distance graph of the Johnson graph $J(n,k)$, provided that $k \leq n/2$. Indeed, vertices of $J(n,k)$ have maximal distance if their intersection size is equal to $k - d = k - \min(k, n-k)$. Be aware that, when k exceeds $n/2$, this number is no longer zero, which means that this definitions is no longer the same as Definition 3.1.

3.1.3 Generalized Johnson graphs

Vertices of Kneser graphs are adjacent if their intersection size is 0. Vertices of Johnson graphs are adjacent if their intersection size is $k-1$. Allowing different intersection sizes between 0 and $k-1$ gives rise to the concept of generalized Johnson graphs, a family of graphs that unites both the Kneser and Johnson graphs.

Definition 3.8

Let $S \subseteq \{0, 1, \ldots, k-1\}$. The **generalized Johnson graph $J_S(n,k)$** has as vertices the k-subsets of $\{1, 2, \ldots, n\}$, where two vertices are adjacent if the intersection size of the corresponding k-subsets is an element of S.

In particular, $J_{\{0\}}(n,k)$ is the Kneser graph $K(n,k)$ and $J_{\{k-1\}}(n,k)$ is the Johnson graph $J(n,k)$. $J_{\{0,2,4,\ldots\}}(n,k)$ is called the *modulo 2 Kneser graph*. Modulo 2 Kneser graphs were introduced in [41] and denoted by $K_2(n,k)$, but we will refrain from that notation, since it would overlap with our notation for the q-Kneser graphs that are defined below.

In the literature, some authors refer to the generalized Johnson graphs as only those graphs $J_S(n,k)$ where S is a singleton $\{i\}$ [4, 38]. In other words, they consider the distance-$(k-i)$ graphs of the Johnson graph (see Theorem 3.7). A common notation for these simpler graphs is $J(n,k,i)$. Even so, we take it a step further and allow multiple intersection sizes for adjacent vertices, similarly to what has been done in [19, 41]. In opposition to the elementary graphs $J_{\{i\}}(n,k)$, our extended definition is closed under complements, which can simplify their treatment.

As the next lemma shows, we may assume that $k \leq n/2$. Instead of writing $\{s+n-2k \mid s \in S\}$, we use the more compact notation $S+n-2k$.

Lemma 3.9

(i) $J_S(n,k) \cong J_{S+n-2k}(n, n-k)$.

(ii) $\overline{J_S(n,k)} = J_{\{0,1,\ldots,k-1\} \setminus S}(n,k)$.

Proof. (i) The map that sends every k-set to its complement in $\{1, 2, \ldots, n\}$ is an isomorphism. Indeed, if two k-sets intersect in an i-set, then their union has size $2k-i$, and the intersection of their complements – which is the complement of their union – is an $n-(2k-i)$-set. Substituting k for $n-k$, we get the converse implication.

(ii) This follows directly from the definition.

□

The generalized Johnson graph $J_S(n,k)$ has $\binom{n}{k}$ vertices and is regular with valency $\sum_{s \in S} \binom{k}{s}\binom{n-k}{k-s}$. In particular, it follows from Theorem 2.20 that the cospectrality of generalized Johnson graphs is the same as that of their complements. Together with Lemma 3.9(ii), this allows us to assume that $|S| \leq k/2$ when proving cospectrality results for generalized Johnson graphs.

3.2 Generalized Grassmann graphs

We continue this chapter by defining q-analogues for the generalized Johnson graphs. Roughly speaking, we replace subsets by subspaces and sizes by dimensions. While the generalized Johnson graphs are studied in combinatorics, the generalized Grassmann graphs belong to the theory of projective geometry. We begin with the subspace versions of Kneser graphs and Johnson graphs – the q-Kneser and Grassmann graphs – which we can then embed in the larger class of generalized Grassmann graphs.

3.2.1 q-Kneser graphs

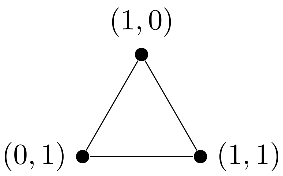
Definition 3.10

The **q-Kneser graph $K_q(n,k)$** has as vertices the k-subspaces of $\mathbb{F}_q^n$, where two vertices are adjacent if the corresponding k-subspaces intersect trivially.

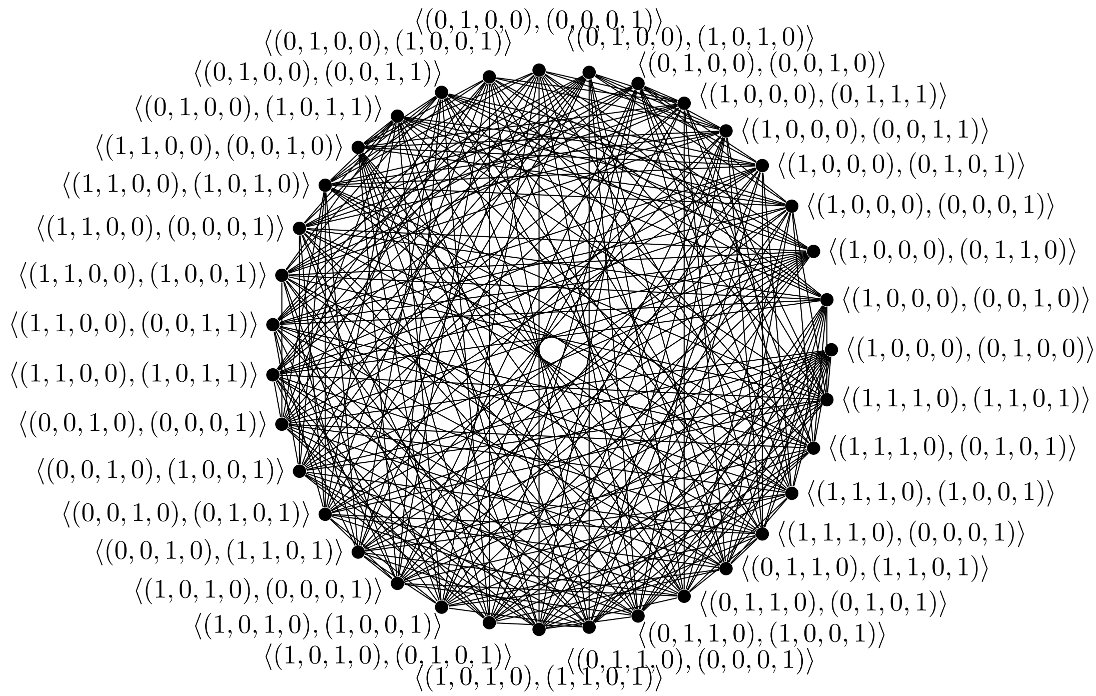
Again, there are some variations on the notation of these graphs in the literature. Some authors write $qK_{n:k}$ or $qK_{n,k}$ instead of $K_q(n,k)$ [18, 22]. Note that when $k=1$ or $k>n/2$, the q-Kneser graphs are trivially complete or edgeless, just like the Kneser graphs. The case $k=n/2$ however, is not trivial.

The number of vertices of $K_q(n,k)$ is equal to $\begin{bmatrix} n \\ k \end{bmatrix}_q$ by Lemma 1.19(i). The neighbours of a vertex v are the k-spaces that intersect v trivially. So $K_q(n,k)$ is regular with valency $q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}_q$ by Lemma 1.19(iii).

Example 3.11. The q-Kneser graph $K_2(2,1)$ has 3 vertices: the points of the projective line $\mathrm{PG}(1,2)$.



Example 3.12. To get an idea of how big q-Kneser graphs can become, consider $K_2(4,2)$, the smallest nontrivial q-Kneser graph. It has $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_2 = 35$ vertices and $\frac{1}{2}\cdot 35\cdot 2^{2^2}\begin{bmatrix} 4-2 \\ 2 \end{bmatrix}_2 = 280$ edges.



The q-Kneser graphs are again edge-regular and even strongly regular if $k = 2$. It is also possible to prove a q-analogue of Theorem 3.4 in a similar way, using Hoffman's ratio bound [35].

3.2.2 Grassmann graphs

Grassmann graphs are probably the most well-known class of graphs in projective geometry. They receive much attention in the literature, partly due to their large role in the theory of error-correcting codes and design theory [36].

Definition 3.13

The **Grassmann graph** or **q-Johnson graph** $\boldsymbol{J_q(n,k)}$ has as vertices the k-subspaces of $\mathbb{F}_q^n$, where two vertices are adjacent if the corresponding k-subspaces intersect in a $(k-1)$-space.

We can prove the following q-analogue of Theorem 3.7.

Theorem 3.14 ([11, Theorem 9.3.3])

The Grassmann graph $J_q(n,k)$ is distance-regular with diameter $d = \min(k, n-k)$ and intersection numbers

$$\begin{aligned} a_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_q \left(q^{i+1} \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q + q \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right), \\ b_i &= q^{2i+1} \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q \begin{bmatrix} n-k-i \\ 1 \end{bmatrix}_q, \\ c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_q^2. \end{aligned}$$

Moreover, we have $\mathrm{d}(v,w) = i \iff \dim(v \cap w) = k - i$.

Proof. We first prove the expression for the distance. It suffices to show that $\mathrm{d}(v,w) \leq i \iff \dim(v \cap w) \geq k - i$. We do this by induction on i. It is trivially true for $i = 0$ and $i = 1$. Now suppose it holds for i, then it must also hold for $i+1$ by the following argument.

If $\mathrm{d}(v,w) \leq i+1$, then there is a k-space u with $\mathrm{d}(u,v) \leq i$ and $u \sim w$. In other words, $\dim(u \cap v) \geq k - i$ and $\dim(u \cap w) = k - 1$. By the Grassmann formula,

$$\begin{aligned} 2k - i - 1 &\leq \dim(u \cap v) + \dim(u \cap w) \\ &= \dim(u \cap v \cap w) + \dim(\langle u \cap v, u \cap w \rangle) \\ &\leq \dim(v \cap w) + \dim(u) \\ &= \dim(v \cap w) + k \end{aligned}$$

so $\dim(v \cap w) \geq k - (i+1)$. Conversely, if $\dim(v \cap w) \geq k - (i+1)$, then we choose a point $p \in v \setminus w$ and a hyperplane π of w through $v \cap w$. The k-space $u := \langle p, \pi \rangle$ intersects v in at least a $(k-i)$-space and w in a $k-1$-space, so $\mathrm{d}(v,w) \leq \mathrm{d}(u,v) + \mathrm{d}(u,w) \leq i+1$.

The intersection dimension of two k-spaces is at least $\max(0, 2k-n)$, so the diameter is $\min(k, n-k)$.

In order to determine the b_i's, consider two k-spaces v and w at distance i and let u be a k-space such that $\mathrm{d}(u,v) = i+1$ and $u \sim w$. Because of the former condition, u does not contain $v \cap w$, but the latter implies it must intersect w in a hyperplane. So there are $\begin{bmatrix} k \\ k-1 \end{bmatrix}_q - \begin{bmatrix} k-(k-i) \\ k-1-(k-i) \end{bmatrix}_q = \begin{bmatrix} k \\ 1 \end{bmatrix}_q - \begin{bmatrix} i \\ 1 \end{bmatrix}_q = q^i \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q$ possibilities for $u \cap w$. There are $\begin{bmatrix} n-(k-1) \\ k-(k-1) \end{bmatrix}_q = \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}_q$ k-spaces through $u \cap w$, but we must avoid the $\begin{bmatrix} k-(k-i-1) \\ k-i-(k-i-1) \end{bmatrix}_q = \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q$ ones intersecting v in a $(k-i)$-space. So there are $\begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} i+1 \\ 1 \end{bmatrix}_q = q^{i+1} \begin{bmatrix} n-k-1 \\ 1 \end{bmatrix}_q$ choices left for u, which results in a total of $b_i = q^{2i+1} \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q \begin{bmatrix} n-k-i \\ 1 \end{bmatrix}_q$.

For the c_i's, consider two k-spaces v and w at distance i and let u be a k-space such that $\mathrm{d}(u,v) = i-1$ and $u \sim w$. By applying the Grassmann formula for $u \cap v$ and $u \cap w$, we see that necessarily $v \cap w \subseteq u$. There are $\begin{bmatrix} k-(k-i) \\ k-i+1-(k-i) \end{bmatrix}_q = \begin{bmatrix} i \\ 1 \end{bmatrix}_q$ choices for $u \cap v$ and $\begin{bmatrix} k-(k-i) \\ k-1-(k-i) \end{bmatrix}_q = \begin{bmatrix} i \\ 1 \end{bmatrix}_q$ choices for $u \cap w$, which makes a total of $c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_q^2$ possible such k-spaces.

The formula for the a_i's follows by applying $b_0 = a_i + b_i + c_i$. □

3.2.3 Generalized Grassmann graphs

Just like before, we can extend the definitions of q-Kneser and q-Johnson graphs to more intersection dimensions. We introduce the following definition, in analogy with the generalized Johnson graphs.

Definition 3.15

Let $S \subseteq \{0, 1, \ldots, k-1\}$. The **generalized Grassmann graph $\boldsymbol{J_{q,S}(n,k)}$** has as vertices the k-subspaces of $\mathbb{F}_q^n$, where two vertices are adjacent if the intersection dimension of the corresponding k-subspaces is an element of S.

In particular, $J_{q,\{0\}}(n,k)$ is the q-Kneser graph $K_q(n,k)$ and $J_{q,\{k-1\}}(n,k)$ is the Grassmann graph $J_q(n,k)$. $J_{q,\{0,2,4,\ldots\}}(n,k)$ is called the *modulo 2 q-Kneser graph*. It is again enough to ask for k to be smaller than $n/2$, thanks to the following lemma.

Lemma 3.16

(i) $J_{q,S}(n,k) \cong J_{q,S+n-2k}(n, n-k)$.

(ii) $\overline{J_{q,S}(n,k)} = J_{q,\{0,1,\ldots,k-1\}\setminus S}(n,k)$.

Proof. (i) A possible isomorphism is given by the map that sends every k-space to its orthogonal complement in $\mathbb{F}_q$. Indeed, consider two k-spaces that intersect in an i-space. Their span has dimension $2k-i$ by Lemma 1.17, so the intersection of their orthogonal complements – which is the orthogonal complement of their span – has dimension $n-(2k-i)$. Substituting k for $n-k$, we get the converse implication.

(ii) By definition. □

The generalized Grassmann graph $J_{q,S}(n,k)$ has $\begin{bmatrix} n \\ k \end{bmatrix}_q$ vertices and is regular with valency

$$\sum_{s \in S} q^{(k-s)^2} \begin{bmatrix} k \\ s \end{bmatrix}_q \begin{bmatrix} n-k \\ k-s \end{bmatrix}_q.$$

Indeed, there are $\begin{bmatrix} k \\ s \end{bmatrix}_q$ possible intersections of size s and another $q^{(k-s)^2}\begin{bmatrix} n-k \\ k-s \end{bmatrix}_q$ possible choices for a $(k-s)$-space in its residue, disjoint with a given $(k-s)$-space (see Lemma 1.19). It follows from Theorem 2.20 and Lemma 3.16(ii) that we may assume $|S| \leq k/2$ when proving cospectrality, just like we could for generalized Johnson graphs.

Remark 3.17. There are more ways to generalize Kneser graphs than using finite vector spaces instead of sets, like we did for the q-Kneser graphs. For example, we could work with trivially intersecting subgroups of certain finite groups, trivially intersecting subspaces of finite polar spaces (see [58]) or flags (informally, collections of nested subspaces) of projective spaces that are in general position (see [23]). However, we will restrict ourselves to the above two families.

3.3 The spectrum of generalized Johnson and Grassmann graphs

This last section is intended as a stepping stone towards Part II. Before we address our main goal of investigating graphs that are cospectral with generalized Johnson and Grassmann graphs, we mention explicit formulas for their eigenvalues, to get an idea of what they look like.

The next two theorems can be found in [10] for S a singleton. Since the eigenvectors are the same for all these graphs (see also [31, Theorem 2.7]), we can just take their sum.

Theorem 3.18 ([10])

Let $k \leq n/2$. The eigenvalues of the generalized Johnson graph $J_S(n,k)$ are given by

$$\lambda_j = \sum_{s \in S} \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} \binom{k-i}{k-s} \binom{n-k+i-j}{n-2k+s}$$

with the corresponding multiplicities $m_j = \binom{n}{j} - \binom{n}{j-1}$, for $j = 0, 1, \ldots, k$.

Theorem 3.19 ([10])

Let $k \leq n/2$. The eigenvalues of the generalized Grassmann graph $J_{q,S}(n,k)$ are given by

$$\lambda_j = \sum_{s \in S} \sum_{i=0}^{j} (-1)^{i+j} q^{(k-s)(k-s+i-j)+\binom{j-i}{2}} \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} k-i \\ k-s \end{bmatrix}_q \begin{bmatrix} n-k+i-j \\ n-2k+s \end{bmatrix}_q$$

with the corresponding multiplicities $m_j = \begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} n \\ j-1 \end{bmatrix}_q$, for $j = 0, 1, \ldots, k$.

Although we will investigate graphs with the above spectra, we do not need these explicit forms to construct graphs that are cospectral with the generalized Johnson and Grassmann graphs when we use the techniques of Chapter 2.

Part II

Cospectrality results

Chapter 4

Overview of old and new results

We have now arrived at the core of this thesis. In this part, we provide various results on the cospectrality of generalized Johnson and Grassmann graphs. Recall that, in order to prove whether these graphs are determined by their spectrum, we can either show that the spectrum determines the graph, or construct cospectral mates. We focus on the latter, by using the techniques introduced in Chapter 2. As we have seen in Chapter 3, the generalized Johnson and Grassmann graphs possess much regularity. This hints at them being good candidates for applying these techniques. More generally, graphs coming from geometries are often ideal candidates for applying switching arguments.

The current chapter serves as an overview of all previously established work, as well as some new discoveries, on the cospectrality of generalized Johnson and Grassmann graphs. Most of these results will be discussed in the following chapters.

4.1 Timeline

Let q be a prime power and assume without loss of generality that $k \leq n/2$ (see Lemma 3.9 and Lemma 3.16).

We present all known results in a chronological order. First, we include two trivial "folklore" facts.

Result 4.1

(i) $J_S(n, 1)$ and $J_{q,S}(n, 1)$ are determined by their spectrum.

(ii) $K(2k, k)$ is determined by its spectrum.

Proof. (i) The vertices of $J_S(n, 1)$ and $J_{q,S}(n, 1)$ are singletons/points, so these graphs are either complete or edgeless. Complete graphs and their complements are DS by Theorem 2.2(iii) (or more directly, using Theorem 1.11 for the edgeless graphs and then Theorem 2.20 for the complete graphs).

(ii) In $K(2k, k)$, every vertex has a unique neighbour: its complement. So $K(2k, k)$ is a disjoint union of complete graphs $K_2 \cup K_2 \cup \cdots \cup K_2$ and therefore DS by Theorem 2.2(iii). □

Since $J_S(n, 1)$ is trivial, we assume $k \geq 2$ from here on.

The first discovery on the cospectrality of generalized Johnson and Grassmann graphs was made independently by Chang and Hoffman [17, 44]. They showed that the triangular graph $T_n = J(n, 2)$ is determined by its spectrum if and only if $n \neq 8$. As a consequence, we have the following equivalent formulation of this finding.

Result 4.2 ([17, 44])

$K(n, 2)$ is determined by its spectrum if and only if $n \neq 8$.

Chang also proved that, up to isomorphism, there are exactly four graphs with the same spectrum as the triangular graph T_8 [16]. In other words (see Example 3.6 and Theorem 2.22), there are exactly four strongly regular graphs with parameters $(28, 12, 6, 4)$. These four graphs are known as the *Chang graphs*.

Another class of graphs that are DS, are the Odd graphs $O_n = K(2n-1, n-1)$. They have been studied by Huang and Liu in [46].

Result 4.3 ([46])

$K(2k+1, k)$ is determined by its spectrum.

Apart from the triangular graphs and the Odd graphs, no other generalized Johnson or Grassmann graphs are known to be DS. This is probably due to the difficulty of proving that graphs are DS, compared to the many ways to construct cospectral mates.

In 2005, van Dam and Koolen constructed a new family of graphs with the same parameters as the Grassmann graphs $J_q(2k+1, k)$, using point-line geometries.

Result 4.4 ([28], see also Theorem 5.15)

$J_q(2k+1, k)$ is not determined by its spectrum.

At about the same time, the above result was extended to many more graphs by van Dam, Haemers, Koolen and Spence [27], who used the same arguments. They were able to formulate what is probably one of the strongest results on the topic yet:

Result 4.5 ([27], see also Corollary 5.6 and Corollary 5.12)

$J(n, k)$ and $J_q(n, k)$ are not determined by their spectrum if $k \geq 3$.

In [41], Haemers and Ramezani constructed cospectral mates for certain Kneser graphs and the modulo 2 Kneser graphs.

Result 4.6 ([41], se also Corollary 6.2 and Theorem 6.3)

(i) $K(n,k)$ is not determined by its spectrum if there is an m such that $2 \leq m \leq k$ and

$$\binom{n-m}{k-m} = 2\binom{n-k-m}{k-m}.$$

(ii) $K_{\{0,2,4,\dots\}}(n,k)$ is not determined by its spectrum if $3 \leq k \leq n-3$.

The most recent previous work on generalized Johnson and Grassmann graphs was done by Cioabă, Haemers, Johnston and McGinnis in [19].

Result 4.7 ([19], see also Theorem 7.3 and Theorem 7.4)

(i) $J_{\{0,1,\dots,m\}}(3k-2m-1,k)$ is not determined by its spectrum if $k \geq \max(m+2,3)$.

(ii) $J_{\{0,1,\dots,m\}}(n,2m+1)$ is not determined by its spectrum if $m \geq 2$ and $n \geq 4m+2$.

Much of the work in this thesis is based on the work by Cioabă, Haemers, Johnston and McGinnis in [19], as it also provides a clear overview of past results. They proved two infinite families of generalized Johnson graphs to be NDS.

In Chapter 8, we discuss the following new finding. It answers the open problem stated in [19].

New result 4.8 (see Corollary 8.5)

$J_{\{2\}}(n,4)$ is not determined by its spectrum.

The above result was found using the recently discovered technique of WQH-switching.

We also prove a result for q-Kneser graphs by GM-switching. The proof can be found in Chapter 9.

New result 4.9 (see Corollary 9.4)

$K_2(n,k)$ is not determined by its spectrum.

If $k=2$, then $K_2(n,2)$ is a strongly regular graph. Indeed, $K_2(n,2)$ is the complement of $J_2(k,2)$ (see Lemma 3.16), which is a strongly regular graph (see Theorem 1.8 and Theorem 3.14). Since strongly regular graphs are characterized by their spectrum (Theorem 2.22), cospectral mates of strongly regular graphs are again strongly regular, with the same parameters. So the construction in Chapter 9 produces new strongly regular graphs with the same parameters as the q-Kneser graphs $K_2(n,2)$.

We end our overview with some "sporadic" cases of graphs that we discovered to be NDS, but are not (yet) proven to belong to a known infinite family of cospectral mates. These graphs were found by computer, by exhaustively going over all possible GM- or WQH-switching sets (up to certain symmetries) that are present in the given graph. We refer to Appendix C for the code.

New result 4.10 (sporadic graphs)

(i) $J_{\{1\}}(11,4)$ is not determined by its spectrum.

(ii) $J_{\{2,4\}}(10,5)$ is not determined by its spectrum.

(iii) $K_3(4,2)$ is not determined by its spectrum.

Proof (by computer). (i) $J_{\{1\}}(11,4)$ has the following two WQH-switching sets of size 3 that produce a nonisomorphic graph.

$$C_1 = \big\{\{1,2,3,10\},\{4,5,6,10\},\{7,8,9,10\}\big\}$$
$$C_2 = \big\{\{1,2,3,11\},\{4,5,6,11\},\{7,8,9,11\}\big\}\,.$$

(ii) $J_{\{2,4\}}(10,5)$ has the following GM-switching set of size 10 that produces a nonisomorphic graph.

$$C = \big\{\{1,2,3,4,7\},\{1,2,3,4,8\},\{3,4,5,6,9\},\{3,4,5,6,10\},\{1,5,6,7,8\},\\ \{2,5,6,7,8\},\{3,7,8,9,10\},\{4,7,8,9,10\},\{1,2,5,9,10\},\{1,2,6,9,10\}\big\}\,.$$

In other words, we take the orbit of the 5-sets $\{1,2,3,4,7\}$ and $\{1,2,3,4,8\}$ under the permutation $(1\;2\;3\;4\;5\;6\;7\;8\;9\;10)^2 = (1\;3\;5\;7\;9)(2\;4\;6\;8\;10)$.

(iii) $K_3(4,2)$ has the following WQH-switching sets of size 3 that produce a nonisomorphic graph.

$$C_1 = \big\{\langle(0,1,0,0),(0,0,0,1)\rangle,\langle(1,1,0,0),(0,0,0,1)\rangle,\langle(1,2,0,0),(0,0,0,1)\rangle\big\}$$
$$C_2 = \big\{\langle(0,0,1,0),(0,0,0,1)\rangle,\langle(1,0,1,0),(0,0,0,1)\rangle,\langle(1,0,2,0),(0,0,0,1)\rangle\big\}\,.$$

This switching set can also be described in a geometric way. Fix two lines L, M that intersect in a point p, and let q be a point that is not in the plane $\langle L, M\rangle$. The vertices of C are the lines spanned by q and a point on either L or M but different from p. □

4.2 Table overview

We can pour the above results into tables to obtain a more structured overview of what is already known and what are new contributions. Each table represents the six smallest generalized Johnson graphs $J_S(n,k)$ or generalized Grassmann graphs $J_{q,S}(n,k)$ for a fixed value of k. The rows are indexed by n, starting from $2k$ (we are assuming that $k \le n/2$). The columns correspond to the different sets S, up to complements, while the last column indicates the number of vertices in the graph (which is independent of S). For the generalized Grassmann graphs, we also separate different values of q. The coloured cells agree with the colours of the results from the previous section. New results are indicated with an asterisk.

For each graph for which the cospectrality is still unknown (the colourless cells), we write down two numbers x, y. In these graphs, we searched exhaustively for possible GM- and WQH-switching sets and found out that there do not exist GM-switching sets of size $\le x$ or WQH-switching sets of size $\le y$ with the property of producing a cospectral mate for the graph. We refer to Appendix C for the code.

Note that similar tables can be found in [19], but we extend those in three ways. First, we add our three new results. Second, our code also checks for WQH-switching sets and is able to produce higher numbers. Third, we include tables for the generalized Grassmann graphs as well.

Legend: Trivial | Hoffman/Chang (1959) | Huang, Liu (1999) | van Dam et al. (2006) | Haemers, Ramezani (2010) | Cioabă et al. (2018) | New result: $J_{\{2\}}(n,4)$ is NDS | Sporadic

$J_S(n,2)$		S: $\{0\}$	$\lvert V\rvert$
n	4	DS	6
	5	DS	10
	6	DS	15
	7	DS	21
	8	NDS	28
	9	DS	36

Table 4.1. Cospectrality of small generalized Johnson graphs with $k = 2$.

$J_S(n,3)$		S			$\lvert V\rvert$
		$\{0\}$	$\{1\}$	$\{2\}$	
n	6	DS	NDS	NDS	20
	7	DS	NDS	NDS	35
	8	NDS	NDS	NDS	56
	9	16, 10	NDS	NDS	84
	10	14, 8	NDS	NDS	120
	11	12, 8	NDS	NDS	165

Table 4.2. Cospectrality of small generalized Johnson graphs with $k = 3$.

$J_S(n,4)$		S							$\lvert V\rvert$
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	
n	8	DS	16, 8	NDS*	NDS	16, 8	NDS	16, 8	70
	9	DS	14, 6	NDS*	NDS	NDS	NDS	14, 6	126
	10	12, 6	12, 6	NDS*	NDS	12, 6	NDS	12, 6	210
	11	NDS	NDS*	NDS*	NDS	10, 6	NDS	10, 6	330
	12	10, 6	10, 6	NDS*	NDS	10, 6	NDS	10, 6	495
	13	10, 6	10, 6	NDS*	NDS	10, 6	NDS	10, 6	715

Table 4.3. Cospectrality of small generalized Johnson graphs with $k = 4$.

Legend: Trivial | Huang, Liu (1999) | van Dam, Koolen (2005) | van Dam et al. (2006) | Haemers, Ramezani (2010) | Cioabă et al. (2018) | New result: $K_2(n,k)$ is NDS | Sporadic

$J_S(n,5)$		S															$\|V\|$
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	$\{0,4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	
n	10	DS	10, 6	10, 6	10, 6	NDS	10, 6	10, 6	10, 6	10, 6	10, 6	NDS	10, 6	10, 6	NDS*	NDS	252
	11	DS	8, 6	8, 6	8, 6	NDS	8, 6	8, 6	8, 6	8, 6	8, 6	NDS	8, 6	8, 6	8, 6	NDS	462
	12	6, 6	6, 6	6, 6	6, 6	NDS	NDS	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	NDS	792
	13	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	NDS	1287
	14	NDS	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	NDS	2002
	15	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	6, 6	6, 6	NDS	6, 6	6, 6	6, 6	NDS	3003

Table 4.4. Cospectrality of small generalized Johnson graphs with $k = 5$.

$J_{q,S}(n,2)$		$q=2$		$q=3$		$q=4$	
		S	$\|V\|$	S	$\|V\|$	S	$\|V\|$
		$\{0\}$		$\{0\}$		$\{0\}$	
n	4	NDS*	35	NDS*	130	4, 4	357
	5	NDS*	155	NDS	1210	NDS	5797
	6	NDS*	651	4, 4	11 011	0, 0	93 093
	7	NDS*	2667	0, 0	99 463	0, 0	$\approx 10^6$
	8	NDS*	10 795	0, 0	$\approx 10^6$	0, 0	$\approx 10^7$
	9	NDS*	43 435	0, 0	$\approx 10^7$	0, 0	$\approx 10^8$

Table 4.5. Cospectrality of small generalized Grassmann graphs with $k = 2$ and $q = 2, 3, 4$.

Legend: van Dam et al. (2006) | New result: $K_2(n,k)$ is NDS

$J_{q,S}(n,3)$		$q=2$				$q=3$				$q=4$			
		S			$\lvert V\rvert$	S			$\lvert V\rvert$	S			$\lvert V\rvert$
		$\{0\}$	$\{1\}$	$\{2\}$		$\{0\}$	$\{1\}$	$\{2\}$		$\{0\}$	$\{1\}$	$\{2\}$	
n	6	NDS*	4, 4	NDS	1395	0, 0	0, 0	NDS	33 880	0, 0	0, 0	NDS	376 805
	7	NDS*	0, 0	NDS	11 811	0, 0	0, 0	NDS	$\approx 10^6$	0, 0	0, 0	NDS	$\approx 10^7$
	8	NDS*	0, 0	NDS	97 155	0, 0	0, 0	NDS	$\approx 10^7$	0, 0	0, 0	NDS	$\approx 10^9$
	9	NDS*	0, 0	NDS	$\approx 10^6$	0, 0	0, 0	NDS	$\approx 10^9$	0, 0	0, 0	NDS	$\approx 10^{11}$
	10	NDS*	0, 0	NDS	$\approx 10^7$	0, 0	0, 0	NDS	$\approx 10^{10}$	0, 0	0, 0	NDS	$\approx 10^{13}$
	11	NDS*	0, 0	NDS	$\approx 10^8$	0, 0	0, 0	NDS	$\approx 10^{11}$	0, 0	0, 0	NDS	$\approx 10^{14}$

Table 4.6. Cospectrality of small generalized Grassmann graphs with $k=3$ and $q=2,3,4$.

$J_{q,S}(n,4)$		$q=2$							
		S							$\lvert V\rvert$
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	
n	8	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	200 787
	9	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^6$
	10	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^8$
	11	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^9$
	12	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{10}$
	13	NDS*	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{11}$

$J_{q,S}(n,4)$		$q=3$							
		S							$\lvert V\rvert$
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	
n	8	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^8$
	9	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{10}$
	10	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{12}$
	11	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{13}$
	12	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{15}$
	13	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{17}$

$J_{q,S}(n,4)$		$q=4$							
		S							$\lvert V\rvert$
		$\{0\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{0,1\}$	$\{0,2\}$	$\{0,3\}$	
n	8	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{10}$
	9	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{12}$
	10	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{14}$
	11	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{17}$
	12	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{19}$
	13	0, 0	0, 0	0, 0	NDS	0, 0	0, 0	0, 0	$\approx 10^{22}$

Table 4.7. Cospectral small generalized Grassmann graphs with $k=4$ and $q=2,3,4$.

Chapter 5

The point-line geometry argument

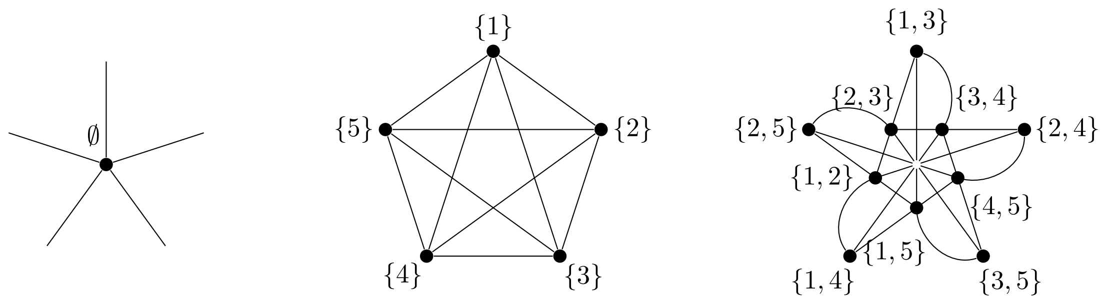
In this chapter, we use point-line geometries to construct cospectral mates for the Johnson and Grassmann graphs with $k \geq 3$, as well as for the Grassmann graph $J_q(5,2)$. It was first proved by van Dam and Koolen that $J_q(2k+1,k)$ is not determined by its spectrum [28], by use of point-line geometries. Not long after, the proof was generalized to almost all (ordinary) Johnson and Grassmann graphs by van Dam, Haemers, Koolen and Spence [27]. Together with Result 4.2, this provides a full determination of the cospectrality of Johnson graphs. For Grassmann graphs, only the cases with $k = 2$ and $n \neq 5$ are still open.

5.1 $J(n,k)$ is not determined by its spectrum if $3 \leq k \leq n-3$

We begin by proving that the Johnson graphs are NDS, except possibly when k is small. The argument, which uses point-line geometries, will also be given in the next section to prove a similar statement for Grassmann graphs. These first two sections do not contain any new results, although we work out the proofs in more detail.

Definition 5.1

Define $\boldsymbol{\mathcal{S}(n,k)}$ as the point-line geometry of which the points are the $(k-1)$-subsets of $\{1,2,\ldots,n\}$, the lines are the k-subsets of $\{1,2,\ldots,n\}$ and where incidence is containment.



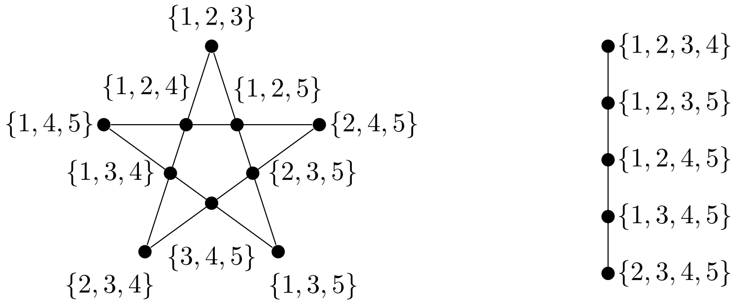


Figure 5.1. The point-line geometries $\mathcal{S}(5, k)$ for $k = 1, 2, 3, 4, 5$.

By taking complements in $\{1, 2, \ldots, n\}$, we observe that $\mathcal{S}(n, k)$ and $\mathcal{S}(n, n-k+1)$ are dual point-line geometries (they can be obtained from each other by reversing the roles of the points and lines), very much like $J(n, k)$ and $J(n, n-k)$ are isomorphic.

The point-line geometry $\mathcal{S}(n, 1)$ consists of one point, $\emptyset$, together with n lines that go through it. Similarly, $\mathcal{S}(n, n)$ has a unique line, on which there are n points. The geometry $\mathcal{S}(n, 2)$ can be considered as the complete graph K_n, where vertices are points and edges are lines. Its dual geometry $\mathcal{S}(n, n-1)$ consists of n pairwise intersecting lines.

The connection between the geometries $\mathcal{S}(n, k)$ and the Johnson graphs is made concrete by the following lemma.

Lemma 5.2 ([27])

$\mathcal{S}(n, k)$ is a partial linear space of order $(k-1, n-k)$ with point graph $J(n, k-1)$ and line graph $J(n, k)$.

Proof. This follows directly from Definition 5.1, together with the fact that the union of two $(k-1)$-sets has size k if and only if the intersection of these sets has size $k-2$. □

We could try to apply Theorem 2.18 on these geometries in order to construct cospectral graphs. The condition that $s = t$ translates into $n = 2k - 1$. However, in this case, the point and line graph are isomorphic. We therefore take another approach. The main idea is to construct a new partial linear space that has the same point graph as $\mathcal{S}(n, k)$, but a different line graph, so we can use Theorem 2.18 to show that these line graphs must be cospectral. We now introduce this new point-line geometry.

Definition 5.3

Let $k \leq n/2 + 1$ and $H := \{1, 2, \ldots, 2k-2\}$. Define $\boldsymbol{\mathcal{S}'(n, k)}$ as the point-line geometry of which the points are the $(k-1)$-subsets of $\{1, 2, \ldots, n\}$ and where there are two types of lines:

(i) k-subsets of $\{1, 2, \ldots, n\}$ that are not fully included in H.

(ii) $(k-2)$-subsets of H.

Incidence is inclusion for the first type of lines, while it is dual inclusion in H for the second type, i.e. a line of the second type is incident with the $(k-1)$-subsets of H that go through it.

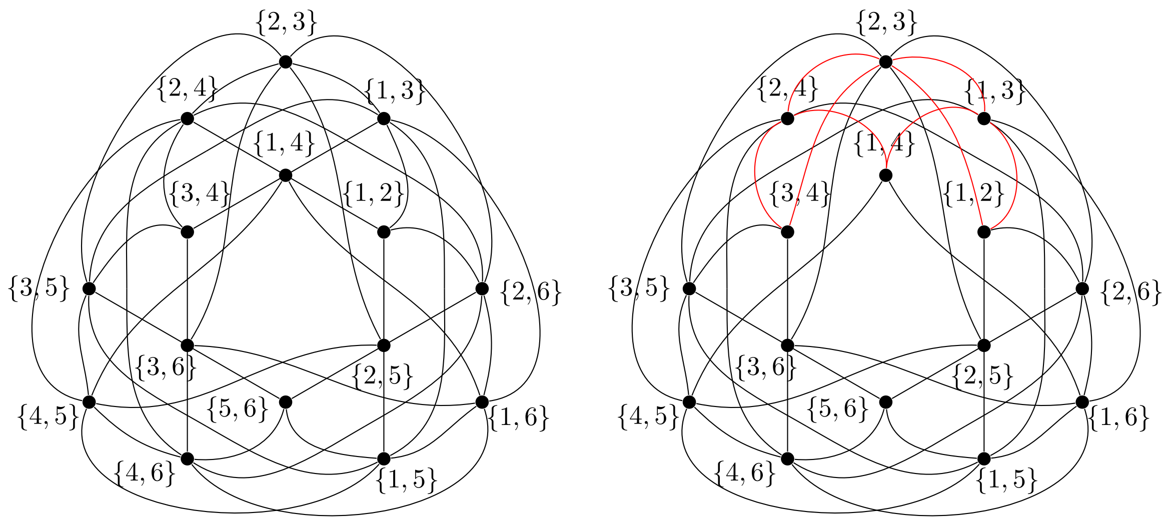


Figure 5.2. The point-line geometries $\mathcal{S}(6,3)$ and $\mathcal{S}'(6,3)$. Lines of the second type are red.

Lemma 5.4 ([27])

$\mathcal{S}'(n,k)$ is a partial linear space of order $(k-1, n-k)$ with point graph $J(n,k-1)$.

Proof. A line of the first type is incident with exactly $\binom{k}{k-1} = k$ points. A line of the second type is incident with a total of $\binom{(2k-2)-(k-2)}{(k-1)-(k-2)} = \binom{k}{1} = k$ points, which is the same amount. A point not fully included in H lies on $\binom{n-(k-1)}{k-(k-1)} = n-k+1$ lines (all of the first type). A point that is included in H, is incident with $\binom{n-(k-1)}{k-(k-1)} - \binom{(2k-2)-(k-1)}{k-(k-1)} = n-2k+2$ lines of the first type and $\binom{k-1}{k-2} = k-1$ lines of the second type, so again, the number of lines through a point is constant.

If two points of $\mathcal{S}'(n,k)$ are collinear, they must intersect in a $(k-2)$-set, since two distinct $(k-1)$-subsets of a given k-set intersect in a $(k-2)$-set. Conversely, consider two $(k-1)$-sets intersecting in all but one element. If these $(k-1)$-sets are both in H, then they lie on a common line of the second type (and their union is no line since it is fully included in H). If not, then they are collinear via a line of the first type: their union. We conclude that the point graph of $\mathcal{S}'(n,k)$ is equal to $J(n,k-1)$. □

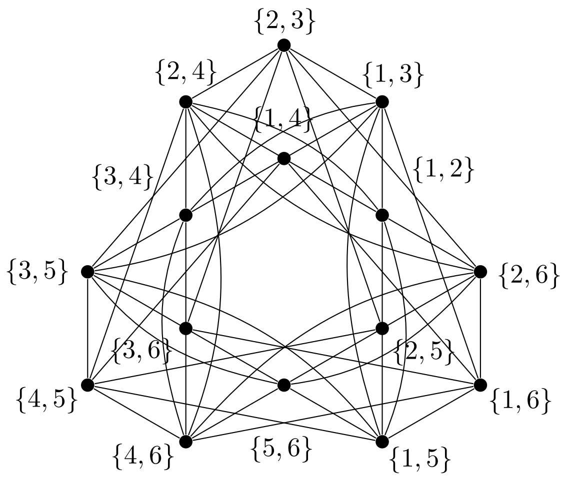


Figure 5.3. The Johnson graph $J(6,2)$ is the point graph of both $\mathcal{S}(6,3)$ and $\mathcal{S}'(6,3)$.

Theorem 5.5 ([27])

Let $3 \leq k \leq n/2$. Then the line graph of $\mathcal{S}'(n,k)$ is not isomorphic to $J(n,k)$.

Proof. Define $L_1 := \{1,2,\ldots,k-2\} \cup \{2k-1,2k\}$ and $L_2 := \{1,2,\ldots,k-2\}$. Then L_1 is a line of the first type and L_2 is a line of the second type in $\mathcal{S}'(n,k)$. In the corresponding line graph, L_1 and L_2 are nonadjacent vertices with $2k$ common neighbours. Indeed, L_1 only has neighbours of the first type and the neighbours of L_2 that are of the first type must intersect H in a set of $k-1$ elements that goes through L_2. There are $\binom{2k-2-(k-2)}{k-1-(k-2)} = k$ choices for this intersection, and 2 choices for the unique element outside H, since it must be an element of $L_1 \setminus H = \{2k-1, 2k\}$.

In $J(n,k)$, nonadjacent vertices have either 0 or 4 common neighbours. Indeed, Theorem 3.7 implies that k-sets at distance 2 intersect in $k-2$ elements. A common neighbour of these k-sets must contain their intersection and one additional element of each of these sets. But there are only 2 additional elements in each set, which results in $2^2 = 4$ choices.

If $k \geq 3$, then $2k \notin \{0,4\}$, so the number of common neighbours of nonadjacent vertices is different for the line graph of $\mathcal{S}'(n,k)$ and the Johnson graph $J(n,k)$. Thus, they are nonisomorphic. □

The original proof of [27] is a bit more general, in the sense that multiple subspaces H are used. However, it suffices to consider just one for obtaining the result.

Corollary 5.6 ([27])

$J(n,k)$ is not determined by its spectrum if $3 \leq k \leq n-3$.

Proof. This follows from Theorem 2.19, Lemma 5.2, Lemma 5.4 and Theorem 5.5. □

5.2 $J_q(n,k)$ is not determined by its spectrum if $3 \leq k \leq n-3$.

We can adapt the same strategy as above to construct a cospectral mate for the Grassmann graph $J_q(n,k)$. We begin by describing the q-analogue of the geometries $\mathcal{S}(n,k)$.

Definition 5.7

Define $\boldsymbol{\mathcal{S}_q(n,k)}$ as the point-line geometry of which the points are the $(k-1)$-subspaces of $\mathbb{F}_q^n$, the lines are the k-subspaces of $\mathbb{F}_q^n$ and where incidence is containment.

Just like before, we see that $\mathcal{S}_q(n,k)$ and $\mathcal{S}_q(n,n-k+1)$ are dual point-line geometries by considering orthogonal complements in $\mathbb{F}_q^n$.

$\mathcal{S}_q(n,1)$ consists of only one point, $\emptyset$, together with $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ lines that go through it. Similarly, $\mathcal{S}(n,n)$ has a unique line, on which there are $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$ points. If $k=2$, then the points and lines of $\mathcal{S}_q(n,k)$ coincide with points and lines of the projective space $\mathrm{PG}(n-1,q)$.

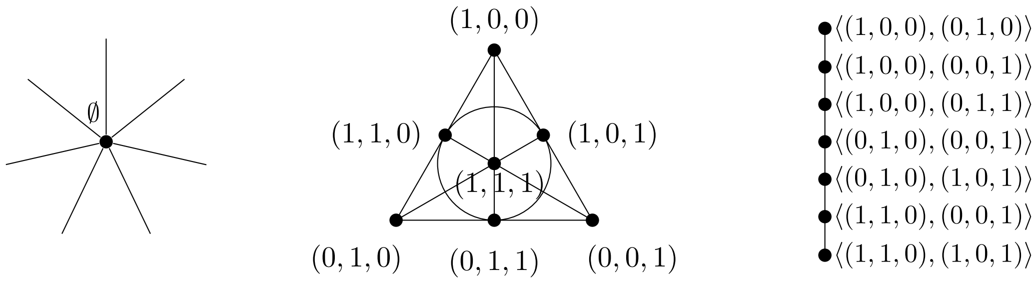


Figure 5.4. The point-line geometries $\mathcal{S}_2(3, k)$ for $k = 1, 2, 3$.

Lemma 5.8 ([27])

$\mathcal{S}_q(n, k)$ is a partial linear space of order $\left(\begin{bmatrix}k\\1\end{bmatrix}_q - 1, \begin{bmatrix}n-k+1\\1\end{bmatrix}_q - 1\right)$ with point graph $J_q(n, k-1)$ and line graph $J_q(n, k)$.

Proof. This follows directly from Definition 5.7, together with the fact that the span of two $(k-1)$-spaces has dimension k if and only if the intersection of these spaces has dimension $k-2$. □

Like before, we define a new, altered point-line geometry that produces a cospectral graph. We write $(\underbrace{*, *, \ldots, *}_{m}, 0, 0, \ldots, 0)$ to denote the subspace $\{(x_1, x_2, \ldots, x_m, 0, 0, \ldots, 0) \mid x_1, x_2 \ldots, x_m \in \mathbb{F}_q\}$.

Definition 5.9

Let $k \leq n/2 + 1$ and $H := (\underbrace{*, *, \ldots, *}_{2k-2}, 0, 0, \ldots, 0)$. Define $\boldsymbol{\mathcal{S}'_q(n, k)}$ as the point-line geometry of which the points are the $(k-1)$-subspaces of $\mathbb{F}_q^n$ and where there are two types of lines:

(i) k-subspaces of $\mathbb{F}_q$ that are not fully included in H.

(ii) $(k-2)$-subspaces of H.

Incidence is inclusion for the first type of lines, while it is dual inclusion in H for the second type, i.e. a line of the second type is incident with the $(k-1)$-subspaces of H that go through it.

Lemma 5.10 ([27])

$\mathcal{S}'_q(n, k)$ is a partial linear space of order $\left(\begin{bmatrix}k\\1\end{bmatrix}_q - 1, \begin{bmatrix}n-k+1\\1\end{bmatrix}_q - 1\right)$ with point graph $J_q(n, k-1)$.

Proof. A line of the first type is incident with exactly $\begin{bmatrix}k\\k-1\end{bmatrix}_q = \begin{bmatrix}k\\1\end{bmatrix}_q$ points. A line of the second type is incident with a total of $\begin{bmatrix}(2k-2)-(k-2)\\(k-1)-(k-2)\end{bmatrix}_q = \begin{bmatrix}k\\1\end{bmatrix}_q$ points, which is the same amount. A point not fully included in H lies on $\begin{bmatrix}n-(k-1)\\k-(k-1)\end{bmatrix}_q = \begin{bmatrix}n-k+1\\1\end{bmatrix}_q$ lines (all of the first type). A point that is included in H, is incident with $\begin{bmatrix} n-(k-1) \\ k-(k-1) \end{bmatrix}_q - \begin{bmatrix} (2k-2)-(k-1) \\ k-(k-1) \end{bmatrix}_q = \begin{bmatrix} n-k+1 \\ 1 \end{bmatrix}_q - \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ lines of the first type and $\begin{bmatrix} k-1 \\ k-2 \end{bmatrix}_q = \begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ lines of the second type, so again, the number of lines through a point is constant.

If two points of $\mathcal{S}'_q(n,k)$ are collinear, they must be $(k-1)$-spaces intersecting in a $(k-2)$-space, since two $(k-1)$-subspaces of a given k-space intersect in at least a $(k-2)$-space. Conversely, consider two $(k-1)$-spaces intersecting in a $(k-2)$-space. If they are both in H, then they lie on a common line of the second type (and their span is no line because it is fully included in H). If not, then they are on a common line of the first type: their span. We conclude that the point graph of $\mathcal{S}'_q(n,k)$ is equal to $J_q(n,k-1)$. □

Theorem 5.11 ([27])

Let $3 \leq k \leq n/2$. Then the line graph of $\mathcal{S}'_q(n,k)$ is not isomorphic to $J_q(n,k)$.

Proof. Define $L_1 := (\underbrace{*,*,\dots,*}_{k-2},\underbrace{0,0,\dots,0}_{k},*,*,0,0,\dots,0)$ and $L_2 := (\underbrace{*,*,\dots,*}_{k-2},0,0,\dots,0)$, then L_1 is a line of the first type and L_2 is a line of the second type in $\mathcal{S}'_q(n,k)$. In the corresponding line graph, L_1 and L_2 are nonadjacent vertices with $(q+1)\begin{bmatrix} k \\ 1 \end{bmatrix}_q$ common neighbours. Indeed, L_1 only has neighbours of the first type and the neighbours of L_2 that are of the first type must intersect H in a $(k-1)$-space that goes through L_2. There are $\begin{bmatrix} 2k-2-(k-2) \\ k-1-(k-2) \end{bmatrix}_q = \begin{bmatrix} k \\ 1 \end{bmatrix}_q$ choices for this intersection, and $\begin{bmatrix} k-(k-2) \\ k-1-(k-2) \end{bmatrix}_q = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = q+1$ choices for the intersection with L_1, since it must be a $(k-1)$-space through L_2 in L_1.

In $J_q(n,k)$, nonadjacent vertices have either 0 or $(q+1)^2$ common neighbours. Indeed, Theorem 3.14 implies that k-spaces at distance 2 intersect in a $(k-2)$-space. A common neighbour of these k-spaces must contain that $(k-2)$-space and intersect each of the k-spaces in dimension $k-1$. There are $\begin{bmatrix} k-(k-2) \\ k-1-(k-2) \end{bmatrix}_q = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = q+1$ ways to do the latter, for both spaces. So there are $(q+1)^2$ choices for the common neighbour.

If $k \geq 3$, then $(q+1)\begin{bmatrix} k \\ 1 \end{bmatrix}_q \notin \{0,(q+1)^2\}$, so the number of common neighbours of nonadjacent vertices is different for the line graph of $\mathcal{S}'_q(n,k)$ and the Grassmann graph $J_q(n,k)$. Thus, both graphs are nonisomorphic. □

Corollary 5.12 ([27])

$J_q(n,k)$ is not determined by its spectrum if $3 \leq k \leq n-3$.

Proof. This follows from Theorem 2.19, Lemma 5.8, Lemma 5.10 and Theorem 5.11. □

5.3 $J_q(2k+1,k)$ is not determined by its spectrum

In [28], van Dam and Koolen proved that the Grassmann graph $J_q(2k+1,k)$ is not determined by its spectrum, using the same technique as above, but with a different argument for nonisomorphism. We give a slightly simplified version of that argument in the proof of the following theorem. First, we need some insight into the cliques of Grassmann graphs.

Lemma 5.13 ([56])

Cliques of $J_q(n,k)$ are k-spaces through a fixed $(k-1)$-space or k-spaces in a fixed $(k+1)$-space.

Proof. Consider an arbitrary clique C in $J_q(n,k)$. We prove the statement by induction on $|C|$.

If $|C| \leq 2$, then we are done. Suppose $|C| = 3$ and let u, v and w be its elements. Since $\dim(u \cap v) = \dim(v \cap w) = k-1$ and $\dim(v) = k$, w must intersect the $(k-1)$-space $u \cap v$ in at least dimension $k-2$ by the Grassmann formula. If $\dim(u \cap v \cap w) = k-1$, then the first option holds. If $\dim(u \cap v \cap w) = k-2$, then $\dim(\langle u \cap w, v \cap w \rangle) = k = \dim(w)$ (again by using the Grassmann formula), so $w \subseteq \langle u, v \rangle$, where $\dim(\langle u, v \rangle) = k+1$.

Now suppose that $|C| \geq 4$ and that the statement holds for any clique of size $|C|-1$. Fix one vertex of the clique. If the other vertices go through a common $(k-1)$-space and this one does not, then all these spaces are in the span of the fixed vertex and any other vertex of C, which is a subspace of dimension $k+1$ (the vertices different from the fixed one form a so-called pencil). On the other hand, if the other vertices lie in a common $(k+1)$-space and the fixed vertex does not, then it must intersect all other k-spaces in the $(k-1)$-space that is its intersection with the bigger $(k+1)$-space (and the vertices different from the fixed vertex form again a pencil). In this case, all vertices of C go through this intersection. In both cases, we are done. □

In particular, we have the following lemma.

Lemma 5.14 ([56])

Maximal cliques in $J_q(n,k)$ consist of all k-spaces through a fixed $(k-1)$-space or all k-spaces in a fixed $(k+1)$-space.

A similar argument shows that the (maximal) cliques of $J(n,k)$ are either k-sets that include a fixed $(k-1)$-set or k-sets in a fixed $(k+1)$-set.

Theorem 5.15 ([28])

$J_q(2k+1,k)$ is not determined by its spectrum.

Proof. It suffices to show that the line graph of $\mathcal{S}'_q(2k+1,k+1)$ is not isomorphic to $J_q(2k+1,k+1) \cong J_q(2k+1,k)$. The result then follows from Theorem 2.19, Lemma 5.8, and Lemma 5.10.

We construct a maximal clique of size $\begin{bmatrix} k+2 \\ 1 \end{bmatrix}_q - 1$ in the line graph of $\mathcal{S}'_q(2k+1,k+1)$. This is enough to prove nonisomorphism, since the maximal cliques of $J_q(2k+1,k+1)$ have size $\begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q$ and $\begin{bmatrix} k+2 \\ 1 \end{bmatrix}_q$ by the previous lemma, which are different numbers. Define the $(k+2)$-space $\pi := (\underbrace{0,0,\dots,0}_{k-1}, \underbrace{*,*,\dots,*}_{k+2})$ that intersects the hyperplane $H = (\underbrace{*,*,\dots,*}_{2k}, 0)$ in a subspace with dimension $k+1$. Let C be the set of $(k+1)$-spaces that are included in π, except for the $(k+1)$-space $\pi \cap H$. Then C is a clique of size $\begin{bmatrix} k+2 \\ 1 \end{bmatrix}_q - 1$. Moreover, it is maximal, since no line of the first type can extend it (otherwise this line would be a vertex in the original graph extending a maximal clique) and no line of the second type can extend it (since no $(k-1)$-space is included in all $(k+1)$-spaces that are included in π). □

The original argument takes into account *all* possible maximal cliques, which allows to prove that the line graph of $\mathcal{S}'_q(2k+1,k+1)$ is not transitive. However, we just want to show that the graph is NDS.

Note that the result only learns us something new about the cospectrality of $J_q(5,2)$, since we already proved the other cases in Corollary 5.12. However, though the result is not as strong as the one in the previous section, it was proved first.

The resulting cospectral graphs are called the *twisted Grassmann graphs* [12]. In [28], it is also proved that these graphs are again distance-regular. Therefore, by combining Theorem 5.15 and Theorem 2.21, we conclude that the twisted Grassmann graphs have the same intersection array as $J_q(2k+1,k)$. Thus, $J_q(2k+1,k)$ is not even determined by its intersection array. In [57], Munemasa proved that these twisted Grassmann graphs can also be obtained from $J_q(2k+1,k)$ by GM-switching. This provides an alternative proof of Theorem 5.15.

5.4 Possible generalizations

The point-line geometries introduced before, allow the following natural generalization.

Definition 5.16

Let $\boldsymbol{\mathcal{S}(n,k,l)}$ be the point-line geometry of which the points are the l-subsets of $\{1,2,\dots,n\}$, the lines are the k-subsets of $\{1,2,\dots,n\}$ and where incidence is containment.

In particular, $\mathcal{S}(n,k) = \mathcal{S}(n,k,k-1)$. The point graph of $\mathcal{S}(n,k,l)$ is now equal to the more general graph $J_{\{2l-k,2l-k+1,\dots,l-1\}}(n,l)$. The line graph of $\mathcal{S}(n,k,l)$ is equal to the graph $J_{\{l,l+1,\dots,k-1\}}(n,k)$.

Unfortunately, $\mathcal{S}(n,k,l)$ is not necessarily a partial space anymore, which means that we cannot adopt Theorem 2.18 or Theorem 2.19. Still, we notice that the graph $J_{\{l,l+1,\dots,k-1\}}(n,k)$ is in fact a "superposition" of the more elementary graphs $J_{\{i\}}(n,k)$, which all have the same eigenvectors (see Section 3.3). In particular, if N is the incidence matrix of $\mathcal{S}(n,k,l)$, we can write N^TN as a linear combination of line matrices of these elementary Johnson graphs $J_{\{i\}}(n,k)$, which all have the same eigenvectors. So, a priori, the line graph of $\mathcal{S}(n,k,l)$ might be cospectral with the line graph of the following geometry.

Definition 5.17

Let $l \leq k \leq 2l \leq n$ and $H := \{1,2,\dots,2l\}$. Define $\boldsymbol{\mathcal{S}'(n,k,l)}$ as the point-line geometry of which the points are the l-subsets of $\{1,2,\dots,n\}$ and where there are two types of lines:

(i) k-subsets of $\{1,2,\dots,n\}$ that are not fully included in H.

(ii) $(2l-k)$-subsets of H.

Incidence is inclusion for the first type of lines, while it is dual inclusion in H for the second type, i.e. a line of the second type is incident with the l-subsets of H that go through it.

One checks that this new geometry also has $J_{\{2l-k,2l-k+1,\dots,l-1\}}(n,l)$ as its point graph. However, the line graph of $\mathcal{S}'(n,k,l)$ is not necessarily cospectral with $J_{\{l,l+1,\dots,k-1\}}(n,k)$. The smallest case in which this goes wrong, is when $(n,k,l) = (7,5,3)$, as was checked by computer. The obtained graph is also not regular anymore. Similar ideas could be tried for the q-analogue case, but here as well, no fruitful results were found.

Chapter 6

Graphs cospectral with Kneser graphs

This chapter is about the results by Haemers and Ramezani in the article [41] with the same name.

The chapter is divided in three sections. First, we discuss a sufficient condition for Kneser graphs to be NDS. Then, we look at a cospectrality result about the modulo 2 Kneser graphs [41] and sketch the proof. We end with some own observations about how these two cospectrality results cannot be extended to q-Kneser graphs.

6.1 Kneser graphs

In the following theorem, we use GM-switching to construct cospectral graphs. In the original paper [41], the constant l is used, whereas we use the constant m that corresponds to $m = k - l$.

Theorem 6.1 ([41, Theorem 2.1])

Let $m \leq k < n/2$ and $M = \{1, 2, \ldots, m\}$. Let C be the set of vertices of $K(n, k)$ that include M. C is a GM-switching set of $K(n, k)$ if and only if

$$\binom{n-m}{k-m} = 2\binom{n-k-m}{k-m}.$$

If moreover $m \geq 2$, then the graph obtained by switching is not isomorphic to $K(n, k)$.

Proof. Since M is not empty, C is a coclique. Its size is $|C| = \binom{n-m}{k-m}$. Choose an arbitrary k-set v not in C. If v and M have a common element, then v has no neighbours in C. If they are disjoint, then v has $\binom{n-k-m}{k-m}$ neighbours in C. So C is a switching set if and only if $\binom{n-k-m}{k-m} = \frac{1}{2}|C|$.

Assume $m \geq 2$ and let $K'(n, q)$ be the graph obtained by switching. We can rewrite the given equation as $\prod_{i=1}^{k-m} \frac{n-k+i}{n-2k+i} = 2$. Since every factor in the product is at least 1, each factor must be smaller or equal than 2. In particular, for $i = 1$, we get $\frac{n-k+1}{n-2k+1} \leq 2$, or $3k - 1 \leq n$, which implies that the diameter of $K(n, k)$ is 2. Indeed, any two intersecting k-sets cover at most $2k - 1$ elements, which leaves at least k elements, so there exists a k-set disjoint with them.

Now define the vertices $v = \{1, 2, \ldots, k\}$ and $w = \{2, 3, \ldots, k+1\}$. Since every vertex of C contains the element 2, w has no neighbours in S. In particular, $v \not\sim w$ both before and after switching. We show that v and w have no common neighbours in $K'(n, k)$, thus proving that the diameter increases after switching and $K(n, k)$ and $K'(n, k)$ are nonisomorphic. Assume, by contradiction, that there exists a k-set u for which $v \sim u \sim w$ in $K'(n,k)$. If $1 \in u$, then u has no neighbours in C, even after switching, contradicting $v \sim u$. So $1 \notin u$. Now both u and w are not in C, so they were originally adjacent in $K(n,k)$ as well, meaning that u is disjoint with w. So $u \subseteq \{k+2, k+3, \ldots, n\}$. So u is also disjoint with M and v, in other words, u has $\frac{1}{2}|C|$ neighbours in C, including v. But then after switching, $v \not\sim u$, a contradiction. □

Corollary 6.2 ([41])

$K(n,k)$ is not determined by its spectrum if there is an m such that $2 \le m \le k$ and

$$\binom{n-m}{k-m} = 2\binom{n-k-m}{k-m}.$$

If $m = k$, the equation becomes $1 = 2$, so there are no solutions. If $m = k-1$, it becomes $n = 3k-1$ and we find the infinite family $K(3k-1,k)$ of graphs that are NDS. If $m = k-2$, we get a solution when $2n = 6k-3+\sqrt{8k^2+1}$. the condition that $\sqrt{8k^2+1}$ is an integer, has infinitely many solutions [41]. If $m \le k-3$, no solutions are known, and it has been conjectured by Erdős that the equation only has a finite number of solutions [32].

6.2 Modulo 2 Kneser graphs

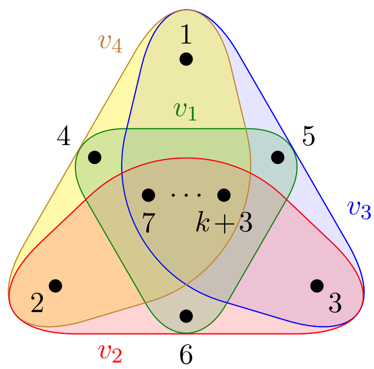
In this section, we have a closer look at the modulo 2 Kneser graphs (see the paragraph below Definition 3.8). The authors of [41] found constructions of graphs that are cospectral but nonisomorphic with all modulo 2 Kneser graphs, except possibly when $k = 2$, in which case the modulo 2 Kneser graphs are the Kneser graphs $K(n,2)$. So together with what we know from Result 4.2, this fully determines the cospectrality of all modulo 2 Kneser graphs graphs.

Theorem 6.3 ([41, Section 3])

$K_{\{0,2,4,\ldots\}}(n,k)$ is not determined by its spectrum if $3 \le k \le n-3$.

Proof (sketch). We construct a cospectral mate by GM-switching. Define $C := \{v_1, v_2, v_3, v_4\}$, where:

- $v_1 := \{1,2,\ldots,k+3\} \setminus \{1,2,3\}$
- $v_2 := \{1,2,\ldots,k+3\} \setminus \{1,4,5\}$
- $v_3 := \{1,2,\ldots,k+3\} \setminus \{2,4,6\}$
- $v_4 := \{1,2,\ldots,k+3\} \setminus \{3,5,6\}$



Every two vertices in C intersect in $k-2$ elements, so C is a clique if k is even and a coclique if k is odd. In any case, C is regular. Now choose an arbitrary vertex $v \notin C$. The elements $1,2,\ldots,6$ are contained in exactly two vertices of C, the elements $7,8,\ldots,k+3$ are contained in all 4 vertices of

C and no vertex of C contains an element that is larger than $k+4$. So $\sum_{i=1}^{4} |v \cap v_i|$ is even, and in particular, v must intersect an even number of vertices of C in an odd number of elements. So the number of neighbours of v in C is even. We conclude that C is a switching set.

It was shown in [41] that the graph obtained by switching is not isomorphic, by checking the number of common neighbours of v_1 and $v_5 := \{1, 2, \ldots, k+3\} \setminus \{4, 5, 6\}$, and in seven cases where this number is the same, by calculating the spectrum of the neighbourhood of v_5 by computer. □

6.3 The constructions fail for q-Kneser and modulo 2 q-Kneser graphs

In this section, we show that the constructions in the previous sections do not generalize to q-Kneser and modulo 2 q-Kneser graphs, at least not in the most natural way. Other cospectral constructions might still exist, but the argument would then be different.

6.3.1 q-Kneser graphs

Let us first look at possible generalizations for the construction of cospectral mates of the Kneser graphs in Theorem 6.1. A problem already occurs in the first part of the theorem, where we establish the equation that guarantees that we are dealing with a switching set.

The most natural q-analogue of the set C in Theorem 6.1 would be as follows. Let M be the fixed m-space $(\underbrace{*, *, \ldots, *}_{m}, 0, 0, \ldots, 0)$ and C the set of vertices of $K_q(n,k)$ that include M.

Again, C is a coclique. Its size is $|C| = \begin{bmatrix} n-m \\ k-m \end{bmatrix}_q$. If any k-space $v \notin C$ has a neighbour in C, it must intersect M trivially. Applying Lemma 1.19(iii) to the residue of M, we find that v has $q^{k(k-m)} \begin{bmatrix} n-k-m \\ k-m \end{bmatrix}_q$ neighbours in C. So if we want C to be a switching set, we need the following condition.

$$\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q = 2q^{k(k-m)} \begin{bmatrix} n-k-m \\ k-m \end{bmatrix}_q$$

Working out the Gaussian coefficients and removing the denominators, we get

$$(q^{n-m}-1)\cdots(q^{n-k+1}-1) = 2q^{k(k-m)}(q^{n-k-m}-1)\cdots(q^{n-2k+1}-1)$$

but modulo q, this becomes $1 \equiv 0$, so the condition is never fulfilled.

6.3.2 Modulo 2 q-Kneser graphs

A natural way to generalize Theorem 6.3 is by mapping each element $i \in \{1, 2, \ldots, n\}$ to the point that has the ith basis vector $\vec{e}_i$ as its homogeneous coordinate, i.e. the point $(0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$ with a 1 on the ith position. In this way, the vertices of the switching set become

- $v_1 := (\underbrace{0, 0, 0, *, *, *}_{6}, \underbrace{*, *, \ldots, *}_{k-3}, 0, 0, \ldots, 0)$
- $v_2 := (\underbrace{0, *, *, 0, 0, *}_{6}, \underbrace{*, *, \ldots, *}_{k-3}, 0, 0, \ldots, 0)$
- $v_3 := (\underbrace{*, 0, *, 0, *, 0}_{6}, \underbrace{*, *, \ldots, *}_{k-3}, 0, 0, \ldots, 0)$
- $v_4 := (\underbrace{*, *, 0, *, 0, 0}_{6}, \underbrace{*, *, \ldots, *}_{k-3}, 0, 0, \ldots, 0)$.

Unfortunately, this is no switching set. For example, the k-space

$$\{(x_1, x_2, \ldots, x_{k+1}, 0, 0, \ldots, 0) \mid x_1, x_2 \ldots, x_{k+1} \in \mathbb{F}_q \text{ and } x_1 = x_2\}$$

intersects v_1 in a $(k-2)$-space and v_2, v_3 and v_4 in a $(k-3)$-space.

The intuition behind this counterexample is that k-spaces are "too big", or that projective spaces are "too sparse": when we are given some pairwise intersecting subspaces, we can almost always find a point that lies on one of them, but not on the others (like the point $(1, 1, 0, 0, \ldots, 0)$ here, which lies on v_4 but not on v_1, v_2 or v_3). So locally, k-spaces through this point do not meet the GM-switching property, and as soon as our total space is large enough, we can extend this point to a k-space that still does not meet the GM-switching property. Compare this to the original switching set, consisting of sets instead of spaces. All the elements are contained in an even number of elements of C, because the sets are "tight". But once we generalize this to spaces, there are too many points.

A more realistic way to generalize the switching set of the proof of Theorem 6.3 is by considering a "tight" constellation of subspaces such that every subspace contains three points. The only candidates for such spaces are lines in a vector space over the finite field $\mathbb{F}_2$. This idea lead to the new result that is worked out in Chapter 9, see also Definition 9.1. The switching set that is introduced in this definition, provides cospectral graphs for the q-Kneser graphs with $q = 2$. For modulo 2 q-Kneser graphs however, it still does not work when $k \geq 3$, because we can consider a k-space through one of the lines, not fully including the other three lines. Since the first line is met in full dimension 2 and every other line in dimension 1, the GM-switching property is violated for modulo 2 Kneser graphs.

In fact, we can rigorously prove that this switching set is not generalizable in any way to a switching set of the same size in (modulo 2) Kneser graphs that works for *arbitrary* q. The reason for this is again the sparsity of projective subspaces, which leads to a lower bound on the size of switching sets. The argument is so general that we devote the following section to it.

6.4 A lower bound on the size of a switching set in generalized Grassmann graphs

In this section, we provide a rough condition to eliminate switching set sizes in the generalized Grassmann graphs $J_{q,S}(n, k)$ where $0 \in S \Leftrightarrow 1 \notin S$. We do this by finding a k-space that intersects one element of the switching set in a point and all others trivially.

Recall from the remarks under Definition 2.12 and Definition 2.14 that the size of a nontrivial switching set is at least 4, because otherwise the graph is left unchanged.

Theorem 6.4

Let $2 \leq k \leq n/2$ and suppose that $0 \in S \Leftrightarrow 1 \notin S$. If $J_{q,S}(n, k)$ has a GM- or WQH-switching set of size at least 4, then the size of this switching set is at least

$$\begin{cases} q+1 & \text{if} \quad k = n/2 \\ q+2 & \text{if} \quad k < n/2. \end{cases}$$

Proof. Assume, by contradiction, that C is a switching set such that $|C| \leq q$ if $k = n/2$ and $|C| \leq q + 1$ if $k < n/2$. In other words, $|C| \leq q + 1$ and $|C| \leq q^{n-2k+1}$.

Fix a k-space $v \in C$ and let π be a $(k+1)$-space through v. The intersection of v with any other k-space has dimension at most $k-1$ and therefore contains at most $\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ points. So at least $\begin{bmatrix} k \\ 1 \end{bmatrix}_q - (|C|-1)\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q$ points are in v but not in any other element of the switching set. Since $|C| \leq q+1$, we have $\begin{bmatrix} k \\ 1 \end{bmatrix}_q - (|C|-1)\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q \geq \begin{bmatrix} k \\ 1 \end{bmatrix}_q - q\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_q = 1$, so there exists at least one such point. Let $p \in v$ be such a point. Then π is a k-space in the residue of p, and so are the $(k+1)$-spaces $\langle p, w \rangle$ for all $w \in C \setminus \{v\}$. Since $|C| \leq q^{(n-1)-k-(k-1)+1}$, we can apply Lemma 1.22 to find a $(k-1)$-space in the residue of p that intersects all these spaces trivially. Back in the full geometry, we have a k-space through p that intersects v in p and every $w \in C \setminus \{v\}$ trivially. Because $0 \in S \Leftrightarrow 1 \notin S$, this k-set is adjacent with v and no other vertex of C, or adjacent all other vertices of C but not with v. In both cases, the switching conditions of both Definition 2.12 and Definition 2.14 are violated when the size of the switching set is at least 4. □

An important corollary of Theorem 6.4 is that, if we want to use switching to construct cospectral graphs for a family of generalized Grassmann graphs where q is unbounded, our switching sets should have variable size. We cannot define a GM-switching set with a fixed number of elements for q-Kneser graphs when q is taken arbitrarily large. For example, there is no switching set of size 4 when $q \geq 4$:

Corollary 6.5

Let $2 \leq k \leq n/2$. The graphs $K_q(n,k)$ and $K_{q,\{0,2,4,\dots\}}(n,k)$ can only contain a GM- or WQH-switching set of size 4 if $q = 2$ or if $q = 3$ and $k = n/2$.

Proof. This follows directly from Theorem 6.4. □

In Chapter 9, we will see that there are indeed switching sets of size 4 in $K_2(n,k)$. In particular, the lower bound $q+2$ in Theorem 6.4 can be reached.

For $q = 3$ and $k = n/2$, we do not know of any switching sets of size 4. More generally, we do not know if the lower bound $q+1$ in Theorem 6.4 is ever reached.

Chapter 7

Cospectral mates for the union of some classes in the Johnson association scheme

In this chapter, we study two results on generalized Johnson graphs that were obtained by Cioabă, Haemers, Johnston and McGinnis in the paper [19] with the same title as the one of this chapter. Two new classes of generalized Johnson graphs were found by applying the technique of GM-switching. The switching sets are similar to the ones used in Chapter 6 for constructing cospectral Kneser graphs, but now, the more general graphs $J_S(n,k)$ with $S = \{0, 1, \ldots, m\}$ are considered. In two cases, the switching proves successful. We devote a section to each one of them. There are many similarities between the proofs, so we first state a definition and a lemma that will be useful for both of them.

Definition 7.1 ([19])

Let Γ be a graph and $v, w \in V(\Gamma)$. The **common neighbour count $\boldsymbol{\lambda(v, w)}$** is the number of common neighbours of v and w. The **common neighbour pattern $\boldsymbol{\Lambda(v)}$** is the multiset of the values $\lambda(v, w)$ for all $w \in V(\Gamma)$.

Lemma 7.2 ([19])

Let Γ be a graph, $C \subseteq V(\Gamma)$ a GM-switching set of Γ and $v, w \notin C$. Then $\lambda(v, w)$ is invariant under GM-switching with respect to C.

Proof. The number of common neighbours of two vertices outside C remains the same, so we only need to focus on the vertices in C. In the following, all adjacencies are taken in Γ. If v or w has no neighbours in C, then we are done. If v or w is adjacent to all vertices of C, then the common neighbours in C are just the neighbours of the other vertex, and this number remains the same as well. If both v and w are adjacent to $\frac{1}{2}|C|$ vertices of C, then

$$\begin{aligned}
|\{u \in C \mid v \sim u \sim w\}| &= |\{u \in C \mid v \sim u\}| - |\{u \in C \mid v \sim u \not\sim w\}| \\
&= \frac{1}{2}|C| - |\{u \in C \mid v \sim u \not\sim w\}| \\
&= |\{u \in C \mid u \not\sim w\}| - |\{u \in C \mid v \sim u \not\sim w\}| \\
&= |\{u \in C \mid v \not\sim u \not\sim w\}|
\end{aligned}$$

The last expression is the number of common neighbours of v and w in C in the new graph, because adjacencies are reversed. □

7.1 $J_{\{0,1,\dots,m\}}(n,k)$ is NDS if $2 \le k = 2m+1 \le n/2$

As usual, we assume $k \ge 2$ to avoid trivialities. In other words, $m \ge 1$ for the m in the title. If $m = 1$, then the statement in the title follows from Corollary 5.6, since $J_{\{0,1\}}(n,3)$ is the complement of $J_{\{2\}}(n,3) = J(n,3)$ and therefore NDS as well (see Theorem 2.20). That is why we will assume $m \ge 2$.

Theorem 7.3 ([19, Section 2.1])

Let $m \ge 2$ with $2m+1 \le n/2$. Then $J_{\{0,1,\dots,m\}}(n, 2m+1)$ is not determined by its spectrum.

Proof. Denote $\Gamma := J_{\{0,1,\dots,m\}}(n, 2m+1)$. Define $M := \{1,2,\dots,2m+2\}$ and let C be the set of vertices of Γ of which the corresponding $(2m+1)$-set is included in M. In other words, $C := \{M \setminus \{x\} \mid x \in M\}$.

We first show that C is a GM-switching set. The vertices of C are $(2m+1)$-sets in a common $(2m+2)$-set, so C has size $|C| = \binom{2m+2}{2m+1} = 2m+2$. Every two $(2m+1)$-sets that are in C, intersect in $2m$ elements, which means that C is a coclique, and therefore regular. Choose an arbitrary vertex $v \notin C$ and let l be the size of its intersection with M. Then v intersects l elements of C in an $(l-1)$-set (those $w \in C$ of which the unique element of $M \setminus w$ is in this intersection) and the other $2m+2-l$ elements in an l-set (those $x \in C$ that fully include this intersection). So if v intersects M in at least $m+2$ elements, then it has no neighbours in C, if v intersects M in exactly $m+1$ elements, then it has $2m+2-(m+1) = \frac{1}{2}|C|$ neighbours, and if v intersects M in at most m elements, then it is adjacent to every vertex of C. We conclude that C is a GM-switching set.

We continue the proof by showing that the graph Γ' obtained from Γ by GM-switching with respect to C, is not isomorphic to Γ. Define $v := \{1,2,\dots,2m+1\} \in C$ and $w := \{2m+2, 2m+3,\dots,4m+2\} \notin C$.

We start by claiming that $\lambda(v,w)$ is different before and after switching. For this, it suffices to count how many of these vertices are lost and how many are added during the GM-switching process. So we only have to consider those neighbours of w that have $\frac{1}{2}|C| = m+1$ neighbours in C. First consider those vertices who are lost. Since these vertices must originally be adjacent to v, they must contain m vertices of v, the unique vertex $2m+2 \in M \setminus v$ and m more vertices outside M that are not all contained in w (otherwise the intersection with w has size $m+1$ and w is not a neighbour). So there are $\binom{2m+1}{m} \cdot \left(\binom{n-(2m+2)}{m} - \binom{2m}{m}\right)$ common neighbours of v and w that are lost during switching. Now consider those vertices that are added. These vertices should not be adjacent to v in the original graph, which means that their intersection with M (which has size $m+1$) must be included in v. In particular, vertices that are added after switching do not contain the element $2m+2$ and therefore, the other m vertices can be chosen freely outside M without the risk of intersecting w in a set that is too large. So there are $\binom{2m+1}{m+1}\binom{n-(2m+2)}{m}$ added common neighbours of v and w, which is $\binom{2m+1}{m}\binom{2m}{m}$ more than the number of lost common neighbours.

Now, assume by contradiction that $\Gamma \cong \Gamma'$. Since Γ is vertex-transitive, all vertices have the same common neighbour pattern. In particular, $\Lambda(w)$ is invariant under switching, i.e. the multiset

$$\Lambda(w) = \{\lambda(u,w) \mid u \in V(\Gamma)\}$$

remains the same before and after switching. By Lemma 7.2, the values of $\lambda(u, w)$ stay the same for vertices $u \notin C$, so we can restrict ourselves to vertices in C, i.e. the multiset

$$\{\lambda(u, w) \mid u \in C\}$$

remains the same. But any two vertices of $C \setminus \{v\}$ can be mapped onto one another by an automorphism that fixes v and w (for example, the automorphism induced by the permutation $(1\,2\,\cdots\,2m{+}1)$ fixes v and w, but shifts the other vertices of C cyclically). So $\lambda(u, w)$ is constant among all $u \in C \setminus \{v\}$, and therefore $\lambda(v, w)$ is the same for Γ and Γ', a contradiction with the above.

We conclude that Γ and Γ' are cospectral mates. Thus, Γ is NDS. □

7.2 $J_{\{0,1,\ldots,m\}}(n, k)$ is NDS if $n = 3k - 2m - 1$, $k \geq m + 2$ and $k \geq 3$

In a similar way, we obtain the following theorem.

Theorem 7.4 ([19])

Let $k \geq \max(2m + 1, 3)$. Then $J_{\{0,1,\ldots,m\}}(3k - 2m - 1, k)$ is not determined by its spectrum.

The proof is almost identical to the one given above, except that the set M is now given by $M := \{1, 2, \ldots, k - 1\}$, C is defined as $C := \big\{M \cup \{x\} \mid x \in \{1, 2, \ldots, n\} \setminus M\big\}$, and the vertices v and w can be chosen as $v := \{1, 2, \ldots, k\} \in C$ and $w := \{2, 3, \ldots, k + 1\} \notin C$.

We refer to the original paper [19] for the full proof.

Chapter 8

$J_{\{2\}}(n, 4)$ is not determined by its spectrum

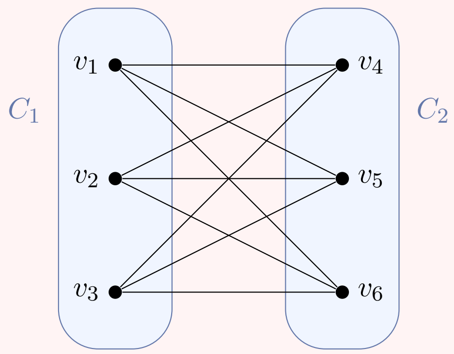
Consider the graph $J_{\{2\}}(n, 4)$, $n \geq 8$. In this chapter, we prove that this graph is not determined by its spectrum, thereby solving the second open problem of [19]. The authors of [19] already discovered that $J_{\{2\}}(8, 4)$ has two cospectral mates by GM-switching with respect to sets of size 8, which means that this graph is NDS. We provide a new infinite family of cospectral mates that extend this cospectrality result to all $n \geq 8$.

Our main tool is WQH-switching. This technique will be applied on a certain switching set of size 6 (more specifically, two sets of size 3) to gain cospectral mates of the studied graphs. The switching set was found by computer for small values of n (the code is given in Appendix C) and turned out to be a valid switching set for all values of n.

Definition 8.1

Define the sets $C_1 := \{v_1, v_2, v_3\}$ and $C_2 := \{v_4, v_5, v_6\}$, where:

- $v_1 := \{1, 2, 3, 4\}$
- $v_2 := \{1, 2, 3, 5\}$
- $v_3 := \{1, 2, 3, 6\}$
- $v_4 := \{1, 4, 5, 6\}$
- $v_5 := \{2, 4, 5, 6\}$
- $v_6 := \{3, 4, 5, 6\}$



Notice the symmetry of the above definition with respect to the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. Indeed, C_1 consists of the 4-sets that include $\{1, 2, 3\}$ and contain one element of $\{4, 5, 6\}$, while C_2 consists of those 4-sets which include $\{4, 5, 6\}$ and contain one element of $\{1, 2, 3\}$.

With this in mind, we can now prove that C_1 and C_2 allow WQH-switching.

Lemma 8.2

C_1 and C_2 form a WQH-switching set of $J_{\{2\}}(n,4)$.

Proof. We check all conditions of Definition 2.14. The first one states that C_1 and C_2 should have the same size, which is clearly the case. Every vertex of C_1 has 0 neighbours in C_1 and 3 neighbours in C_2, and every vertex of C_2 has 0 neighbours in C_2 and 3 neighbours in C_1, so Definition 2.14(ii) is also fulfilled ($c = -3$). In order to check the third condition, select an arbitrary 4-set $v \notin C_1 \cup C_2$. We distinguish seven cases, according to the intersection size of v with $\{1,2,3\}$ and $\{4,5,6\}$, see Table 8.1. Because of the symmetry of C with respect to these sets, we may assume that $|v \cap \{1,2,3\}| \leq |v \cap \{4,5,6\}|$. Also note that $|v\cap\{1,2,3\}|+|v\cap\{4,5,6\}| \leq 4$, and that these cardinalities cannot be 1 and 3 simultaneously, since otherwise the vertex is one of the vertices in $C_1 \cup C_2$. In each case, the requirements of Definition 2.14(iii) are met. ☐

$\|v \cap \{1,2,3\}\|$	0	0	0	0	1	1	2
$\|v \cap \{4,5,6\}\|$	0	1	2	3	1	2	2
# neighbours in C_1	0	0	0	0	1	2	1
# neighbours in C_2	0	0	3	0	1	2	1

Table 8.1. The number of neighbours of v in C_1 and C_2, depending on its intersection size with $\{1,2,3\}$ and $\{4,5,6\}$.

We are left to prove that the graph obtained by switching with respect to (C_1, C_2), is not isomorphic to the original one. We will use the fact that $J_{\{2\}}(n,4)$ is edge-regular, while the new graph is not. Actually, every "elementary" generalized Johnson graph $J_{\{i\}}(n,k)$ is edge-regular, because $J_{\{i\}}(n,k)$ is the distance-$(k-i)$ graph of the Johnson graph $J(n,k)$, which is distance-regular (see Theorem 3.7). However, we give a direct proof of this particular case, since we also need to know the parameters of the graph.

Lemma 8.3

$J_{\{2\}}(n,4)$ is edge-regular with parameters $\left(\binom{n}{4}, 6\binom{n-4}{2}, \frac{1}{2}n(n+3)-26\right)$.

Proof. The vertices of $J_{\{2\}}(n,4)$ are the subsets of $\{1,2,\ldots,n\}$ of size 4, so there are $\binom{n}{4}$ vertices. Choose an arbitrary vertex v. Neighbours of v contain 2 elements in v and 2 elements outside v. So v has $\binom{4}{2}\binom{n-4}{2}$ neighbours. Choose a second vertex w that is adjacent with v. By relabelling the elements of $\{1,2,\ldots,n\}$, we may assume that $v = \{1,2,3,4\}$ and $w = \{1,2,5,6\}$. There are three types of common neighbours.

(i) Vertices that include $\{1,2\}$. The other two elements can be chosen from $\{7,8,\ldots,n\}$. So there are $\binom{n-6}{2}$ such vertices.

(ii) Vertices that intersect $\{1,2\}$ in one element. They must contain one element of $\{3,4\}$ and one element of $\{5,6\}$, while the remaining element should be in $\{7,8,\ldots,n\}$. There are $2^3(n-6)$ such vertices.

(iii) If a vertex is adjacent with v and w but does not contain 1 or 2, it must be equal to $\{3,4,5,6\}$.

We conclude that v and w have $\binom{n-6}{2} + 2^3(n-6) + 1 = \frac{1}{2}n(n+3) - 26$ common neighbours. ☐

We are now ready to prove the main theorem of this chapter.

Theorem 8.4

Let $n \geq 8$ and $\Gamma := J_{\{2\}}(n, 4)$. let Γ' be the graph obtained from Γ by WQH-switching with respect to (C_1, C_2). Then Γ and Γ' are nonisomorphic.

Proof. Consider the vertices $v = \{1, 2, 3, 4\} \in C$ and $w = \{1, 4, 5, 7\} \notin C$. They are adjacent in both Γ and Γ', since $|w \cap \{1, 2, 3\}| = 1$ and $|w \cap \{4, 5, 6\}| = 2$ (see Table 8.1). We show that they have more common neighbours in Γ' than in Γ. For this, we only need to consider neighbours of w for which the adjacency with v is changed during the switching process. We observe from Table 8.1 that the switching only affects those vertices which have 2 elements in one of the sets $\{1, 2, 3\}$, $\{4, 5, 6\}$ and none in the other. Neighbours of w that contain 2 elements of $\{1, 2, 3\}$ and none of $\{4, 5, 6\}$, are of the form $\{1, 2, 7, a\}$ or $\{1, 3, 7, a\}$ with $a \geq 8$. So there are a total of $2(n-7)$ vertices that are adjacent with v in Γ but not in Γ'. On the other hand, neighbours of w that contain no element of $\{1, 2, 3\}$ and 2 elements of $\{4, 5, 6\}$, are of the form $\{4, 5, a, b\}$, $\{4, 6, 7, a\}$ or $\{5, 6, 7, a\}$ with $a, b \geq 8$. They are nonadjacent with v in Γ, but become adjacent with v after switching. So $\binom{n-7}{2} + 2(n-7)$ new common neighbours are created.

We conclude that v and w have $\binom{n-7}{2}$ more common neighbours in Γ' than any two adjacent vertices in Γ, which is a strictly positive difference, except when $n = 8$. In order to solve this one specific case, we take a look at an other pair of vertices.

Assume $n = 8$ and consider the vertices $v = \{1, 2, 3, 4\} \in C$ and $u = \{5, 6, 7, 8\} \notin C$. They are nonadjacent in Γ, but become adjacent after switching, since $|u \cap \{1, 2, 3\}| = 0$ and $|u \cap \{4, 5, 6\}| = 2$ (see Table 8.1). We give a lower bound on the number of common neighbours of v and u in Γ'. In the original graph, v and u have $\binom{4}{2}^2 = 36$ common neighbours. Three of these are the vertices of C_2, and are therefore lost after switching. All other common neighbours of v and u in Γ that are no longer common neighbours in Γ', are those that contain 2 elements of $\{1, 2, 3\}$ and none of $\{4, 5, 6\}$, i.e. the three vertices $\{1, 2, 7, 8\}$, $\{1, 3, 7, 8\}$ and $\{2, 3, 7, 8\}$. We are left with at least $36 - 3 - 3 = 30$ common neighbours (and possibly more, since other common neighbours might be added after switching), which is strictly more than $\frac{1}{2}8(8+3) - 26 = 18$.

We proved that there are always two neighbours in Γ' which have strictly more common neighbours than any two neighbours in Γ. So Γ and Γ' cannot be isomorphic. ☐

Note that there are still many neighbours in the new graph with the same number of common neighbours as in the original graph, by a somewhat similar (but easier) argument as in the proof of Lemma 7.2 (in this way, we could prove that the new graph is not edge-regular).

Corollary 8.5

$J_{\{2\}}(n, 4)$ is not determined by its spectrum if $n \geq 8$.

Proof. This follows from Theorem 2.16 and Theorem 8.4. ☐

Chapter 9

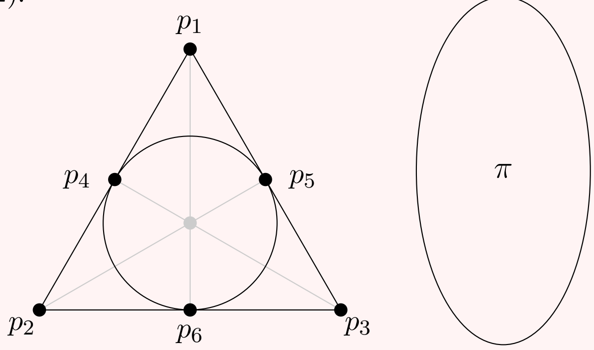
q-Kneser graphs are not determined by their spectrum if $q = 2$

In this chapter, we present a new result on the cospectrality of q-Kneser graphs with $q = 2$. This is done by GM-switching. The switching set that is used, was found by computer for the graph $K_2(4, 2)$ and proved to be extendable to all values of n and k. We introduce this set in the following definition. We use a similar coordinatization to the one in Example 1.16. As usual, we assume that $2 \leq k \leq n/2$.

Definition 9.1

Define the following points in $PG(n-1, 2)$.

- $p_1 := (1, 0, 0, 0, 0, \dots, 0)$
- $p_2 := (0, 1, 0, 0, 0, \dots, 0)$
- $p_3 := (0, 0, 1, 0, 0, \dots, 0)$
- $p_4 := (1, 1, 0, 0, 0, \dots, 0)$
- $p_5 := (1, 0, 1, 0, 0, \dots, 0)$
- $p_6 := (0, 1, 1, 0, 0, \dots, 0)$



Define the $(k-2)$-space $\pi := (0, 0, 0, \underbrace{*, *, \dots, *}_{k-2}, 0, 0, \dots, 0)$ and the set $C := \{p_1p_2\pi,\ p_1p_3\pi,\ p_2p_3\pi,\ p_4p_5\pi\}$ of k-spaces.

Lemma 9.2

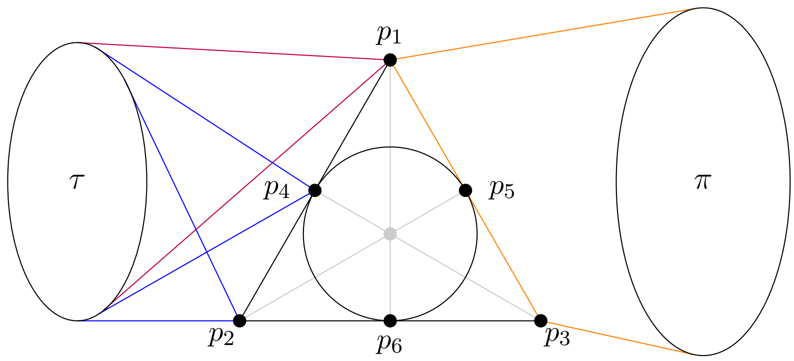
C is a GM-switching set of $K_2(n, k)$.

Proof. Every element of C meets every other element of C, so C is a coclique and thus it is regular. Consider an arbitrary k-space not in C and let R be the set $\{p_1\pi, p_2\pi, \dots, p_6\pi\}$ containing "residual points of π". If the chosen k-space intersects all elements of R trivially, then it also intersects all spaces in C trivially, so it is adjacent to all the corresponding elements of the q-Kneser graph. If it intersects just one of the $(k-1)$-spaces in R, then it is adjacent to exactly 2 elements of C. And if it intersects at least two elements of R, then it meets every element of C because it contains a line in the $k+1$-space $p_1p_2p_3\pi$, therefore being adjacent to no element of C. $\square$

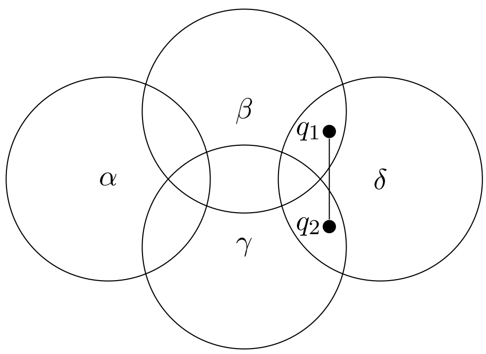
Theorem 9.3

Let $2 \leq k \leq n/2$ and $\Gamma := K_2(n,k)$. Let Γ' be the graph obtained from Γ by GM-switching with respect to C. Then Γ and Γ' are nonisomorphic.

Proof. First choose a $(k-1)$-space τ that intersects $p_1p_2p_3\pi$ trivially and consider the pairwise nonadjacent k-spaces $p_1\tau$, $p_2\tau$ and $p_4\tau$ in Γ'. Then there is exactly one k-space that is adjacent to $p_1\tau$ but nonadjacent to both $p_2\tau$ and $p_4\tau$. Indeed, a k-space outside C that intersects both $p_2\tau$ and $p_4\tau$ intersects $p_1\tau$ as well. So the only spaces that can meet this property are in C. Since the switching reverses adjacency for $p_1\tau$, $p_2\tau$ and $p_4\tau$, we get that $p_1p_3\pi$ is adjacent to $p_1\tau$ and nonadjacent to $p_2\tau$ and $p_4\tau$, while the other elements of C are not.



Now consider three arbitrarily chosen nonadjacent k-spaces in Γ. Call them α, β and γ. Nonadjacency here means that they intersect one another. We prove that the number of k-spaces that intersects α trivially but intersects both β and γ nontrivially, is never equal to one. Let δ be such a space (if this does not exist, we are done). Choose $q_1 \in \beta \cap \delta$ and $q_2 \in \gamma \cap \delta$. The line q_1q_2 does not intersect α (it lies in δ). Now we can easily construct multiple k-spaces that go through q_1q_2 and intersect α trivially (to be exact, there are $2^{k(k-2)} \begin{bmatrix} n-k-1 \\ k-2 \end{bmatrix}_2$ such spaces). But then these spaces meet the above property, and we already found more than one.



Since the number of k-spaces with this property is different for Γ and Γ', while it should be invariant under isomorphism, we conclude that Γ and Γ' are not isomorphic. □

Corollary 9.4

Let $2 \leq k \leq n/2$. Then $K_2(n,k)$ is not determined by its spectrum.

Proof. This follows from Theorem 2.11 and Theorem 9.3. □

Part III

Other results

Chapter 10

Diameter and girth of generalized Johnson and Grassmann graphs

In this last part of the thesis, we have a look at some other interesting results. The aim of this chapter is to find two related metrics of the generalized Johnson and Grassmann graphs: the diameter and the girth. Although the diameter is a structural invariant, we can relate it to the eigenvalues. For example, the diameter is always smaller than the number of distinct eigenvalues. There also exist stronger bounds on the diameter in terms of the actual eigenvalues [24].

The diameter of the ordinary Kneser graph has already been established in [62] and was generalized further to a large group of generalized Johnson graphs in [4], while the latter also contains an expression for the girth. The aim of this chapter is to extend these results to the q-analogue case.

10.1 Generalized Johnson graphs

Recall from Definition 1.5 that the diameter of a graph Γ is denoted by $\operatorname{diam}(\Gamma)$, and the girth by $\mathrm{g}(\Gamma)$. Just like before, the assumption $k \leq n/2$ can be made without loss of generality. The following theorem demands the slightly stronger condition $k < n/2$, because if $k = n/2$, then the Kneser graph $K(n,k)$ is not connected.

Theorem 10.1 ([62])

Let $k < n/2$. The Kneser graph $K(n,k)$ has diameter $\left\lceil \frac{k-1}{n-2k} \right\rceil + 1$.

Theorem 10.2 ([4, Theorem 4.2])

Let $k \leq n/2$ and $s < k$ with $(n,k,s) \neq (2k,k,0)$. The generalized Johnson graph $J_{\{s\}}(n,k)$ has diameter

$$\operatorname{diam}\left(J_{\{s\}}(n,k)\right) = \begin{cases} \left\lceil \frac{k-s-1}{n-2k+2s} \right\rceil + 1 & \text{if } n < 3(k-s)-1 \text{ or } s = 0 \\ 3 & \text{if } 3(k-s)-1 \leq n < 3k-2s \text{ and } s \neq 0 \\ \left\lceil \frac{k}{k-s} \right\rceil & \text{if } 3k-2s \leq n \text{ and } s \neq 0. \end{cases}$$

Theorem 10.3 ([4, Theorem 2.4])

Let $k \leq n/2$ and $s < k$ with $(n, k, s) \neq (2k, k, 0)$. The generalized Johnson graph $J_{\{s\}}(n, k)$ has girth

$$g\left(J_{\{s\}}(n,k)\right) = \begin{cases} 3 & \text{if } 3(k-s) \leq n \\ 4 & \text{if } n < 3(k-s) \text{ and } (n,k,s) \neq (2k+1,k,0) \\ 5 & \text{if } (n,k,s) = (5,2,0) \\ 6 & \text{if } (n,k,s) = (2k+1,k,0) \text{ and } k \geq 3. \end{cases}$$

Proving the above two theorems is not a trivial task, as one can expect from the case distinction in the statements. Generalizing this to arbitrary sets $S \subseteq \{0, 1, \ldots, k-1\}$ becomes even more complicated, so we will not attempt in doing so.

10.2 Generalized Grassmann graphs

It is actually easier to determine the girth and diameter of generalized Grassmann graphs than those of the generalized Johnson graphs. This is due to the fact that there are many more mutually trivially intersecting k-subspaces of an n-space than there are mutually disjoint k-subsets of an n-set. This was made concrete in Lemma 1.22.

We are now ready to state the q-analogues of Theorem 10.2 and Theorem 10.3. We begin with the girth, since it has the simplest expression.

Theorem 10.4

Every generalized Grassmann graph $J_{q,S}(n, k)$ with $S \neq \emptyset$ has girth 3.

Proof. Let $J_{q,S}(n, k)$ be a nontrivial Grassmann graph and let $s \in S$. Recall from Lemma 3.16 that we may assume that $k \leq n/2$. Choose two k-spaces v and w that intersect in an s-space π. Since $2 \leq q \leq q^{n-2k+1} \leq q^{(n-s)-2(k-s)+1}$, we can apply Lemma 1.22 to the residual projective space of π to find a third k-space u that intersects v and w in π. Then u, v and w are mutually adjacent, i.e. there is a triangle in the graph. We conclude that the girth must be 3. □

Theorem 10.5

Let $k \leq n/2$ and suppose that S is a nontrivial subset of $\{0, 1, \ldots, k-1\}$ with least element s. The generalized Grassmann graph $J_{q,S}(n, k)$ has diameter

$$\operatorname{diam}\left(J_{q,S}(n,k)\right) = \begin{cases} 2 & \text{if } s = 0 \\ \left\lceil \frac{k}{k-s} \right\rceil & \text{if } s \neq 0. \end{cases}$$

Proof. Let v and w be two arbitrary vertices with intersection dimension t. We show that

$$d(v,w) = \begin{cases} \left\lceil \frac{k-t}{k-s} \right\rceil & \text{if } t < s \\ 1 \text{ or } 2 & \text{if } t \geq s. \end{cases}$$

The result then follows because $\left\lceil \frac{k-t}{k-s} \right\rceil$ is maximal for $t = 0$. If $s \neq 0$, then $\left\lceil \frac{k}{k-s} \right\rceil$ is greater than 2 and it really is the maximum distance. But if $s = 0$, then we always have $t \geq s$ and the diameter is 2 because S is nontrivial.

Case 1: $t < s$. First assume $t < s$. Then in particular, $v \not\sim w$. We must prove that $\mathrm{d}(v, w) = d'$, where $d' := \left\lceil \frac{k-t}{k-s} \right\rceil$. We distinguish two cases, similarly to the proof of [4, Lemma 3.2].

Case 1(a): $2s \leq k + t$. In this case, $k - s < k - t \leq 2(k - s)$, so $d' = 2$. It suffices to construct a k-space that intersects both v and w in an s-space. Choose an s-space π through $v \cap w$ in v and an s-space τ through $v \cap w$ in w. These two span a $2s - t$-space, which we can extend to a k-space that intersects v and w in just this space, by applying Lemma 1.22 to its residue. So $\mathrm{d}(v, w) = 2 = d'$.

Case 1(b): $k + t < 2s$. We first show that $\mathrm{d}(v, w) \leq d'$ by constructing a walk of length d' from v to w. Choose a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_t\}$ of $v \cap w$ and expand it to a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_t, \vec{y}_1, \vec{y}_2, \ldots, \vec{y}_{k-t}\}$ of v and to a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_t, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{k-t}\}$ of w. Define

$$u_i := \langle \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_t, \vec{y}_{i(k-s)+1}, \vec{y}_{i(k-s)+2}, \ldots, \vec{y}_{k-t}, \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_{i(k-s)} \rangle$$

for $i \in \{1, 2, \ldots, d' - 2\}$. Then $(v, u_1, u_2, \ldots, u_{d'-2})$ is a walk of length $d' - 2$. The intersection of $u_{d'-2}$ and w is at least $2s - k$, so by applying the first case, we can extend the walk to w by only two more steps, which results in a walk of length d' between v and w.

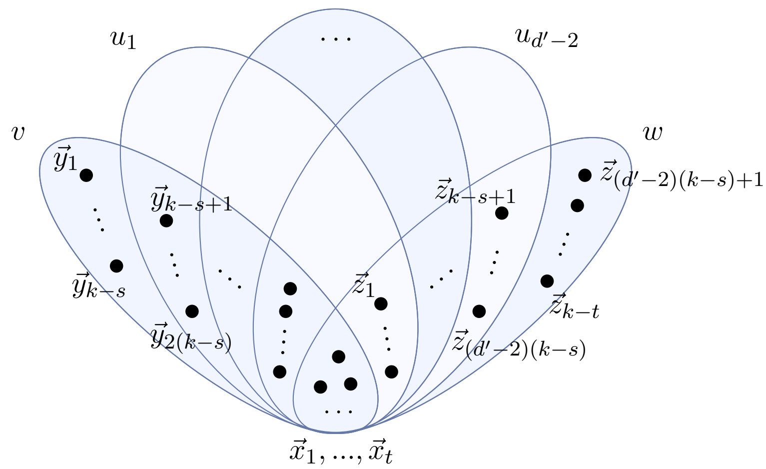


Figure 10.1. The walk $(v, u_1, u_2, \ldots, u_{d'-2})$.

In order to show the converse inequality $\mathrm{d}(v, w) \geq d'$, it suffices to prove that $\mathrm{d}(v, w) \geq \frac{k-t}{k-s}$, since $\mathrm{d}(v, w)$ is an integer. We do this by induction on $\mathrm{d}(v, w)$. If $\mathrm{d}(v, w) = 1$, then $\mathrm{d}(v, w) \geq \frac{k-t}{k-s}$ simply because $s \leq t$. For the induction step, consider a vertex u with $\mathrm{d}(u, v) = \mathrm{d}(v, w) - 1$ and $u \sim w$. The induction hypothesis implies that $\mathrm{d}(u, v) \geq \frac{k - \dim(u \cap v)}{k-s}$, or after rewriting, $(s-k) \cdot \mathrm{d}(v, w) - s + 2k \leq \dim(u \cap v)$. Thus,

$$\begin{aligned}
(s - k) \cdot \mathrm{d}(v, w) + 2k &\leq \dim(u \cap v) + s \\
&\leq \dim(u \cap v) + \dim(u \cap w) \\
&= \dim(u \cap v \cap w) + \dim(u \cap \langle v, w \rangle) \\
&\leq \dim(v \cap w) + \dim(u) \\
&= t + k
\end{aligned}$$

where we used that $u \sim w$ in the second step, and applied the Grassmann formula (Lemma 1.17) in the third step. We conclude that $\mathrm{d}(v, w) \geq \frac{k-t}{k-s}$.

Case 2: $t \geq s$. If $t \geq s$, we can choose an s-space π in the intersection of v and w and construct a k-space u that intersects v and w in π using Lemma 1.22, like we did in Case 1. This construction provides us with a walk of length 2, so the distance $\mathrm{d}(v, w)$ is at most 2. □

Allowing k to be larger than $n/2$ leads to the following, more general statement. Note that the condition $2k \leq n + \max(S)$ below is just there to ensure connectivity. Indeed, by the Grassmann formula (Lemma 1.17), the intersection dimension of two k-spaces is at least $2k - n$. So S must contain an element that is at least $2k - n$. Otherwise, the graph is edgeless.

Corollary 10.6

Let S be a nontrivial subset of $\{0, 1, \ldots, k-1\}$ with least element s such that $2k \leq n + \max(S)$. The generalized Grassmann graph $J_{q,S}(n, k)$ has diameter

$$\operatorname{diam}\left(J_{q,S}(n,k)\right) = \begin{cases} 2 & \text{if } s \in \{0, 2k-n\} \\ \left\lceil \frac{\min(k, n-k)}{k-s} \right\rceil & \text{if } s \notin \{0, 2k-n\}. \end{cases}$$

Proof. We distinguish two cases.

Case 1: $k \leq n/2$. We are done by the previous theorem.

Case 2: $k > n/2$. We know from Lemma 3.16 that $J_{q,S}(n,k) \cong J_{q,S+n-2k}(n, n-k)$. Applying the first case on the latter graph (since $2(n-k) < n$ and $S \subseteq \{0, 1, \ldots, n-k-1\}$), results in the diameter being $\left\lfloor \frac{n-k}{n-k-(s+n-2k)} \right\rfloor = \left\lfloor \frac{n-k}{k-s} \right\rfloor$ if $s + n - 2k \neq 0$ and 2 if $s + n - 2k = 0$. □

Chapter 11

A lower bound on the number of graphs that are not determined by their spectrum

In this chapter, we give an asymptotic lower bound on the number of graphs that are NDS. We focus on graphs that are cospectral through GM-switching with respect to a switching set of size 4, because there is computational evidence that this is the most productive switching size [42, Tables 1 and 2]. This lower bound has been mentioned several times in the literature [26, 37, 42], but never been proved rigorously. We aim to settle this by working out the details.

The main idea of the proof is to calculate the proportion of graphs that are equipped with a switching set, together with the observation that most of these graphs are *asymmetric* enough such that the graph obtained by GM-switching is nonisomorphic:

Definition 11.1

A graph is **symmetric** if it has a nontrivial automorphism. Otherwise, it is **asymmetric**.

Be aware that some authors define symmetry as being "vertex and edge transitive", which is a stronger requirement [38].

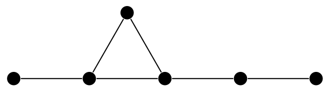
We use the following nontrivial result by Erdős and Rényi (though the original statement is stronger and more formal).

Theorem 11.2 ([34, Theorem 2])

Almost all graphs are asymmetric, i.e. the proportion of symmetric graphs on n vertices goes to 0 as $n \to \infty$.

We use the asymptotic notation $o(1)$ for quantities that go to zero if $n \to \infty$. So the statement is equivalent to saying that the number of symmetric graphs is $2^{\binom{n}{2}} o(1)$, and that the number of asymmetric graphs is $2^{\binom{n}{2}}(1 - o(1))$.

Example 11.3. Although most graphs are asymmetric, many *small* graphs happen to be symmetric. The smallest examples of asymmetric graphs have 6 vertices. Up to isomorphism, there are eight such graphs, one of which is given below.



Note that symmetry is preserved when we take the complement of a graph, since an automorphism of a graph is also an automorphism of its complement.

We already observed that there are $2^{\binom{n}{2}}$ (possibly isomorphic) graphs on n vertices. The number of nonisomorphic graphs on n vertices is smaller:

Corollary 11.4

The number of nonisomorphic graphs on n vertices is $\frac{1}{n!}2^{\binom{n}{2}}(1 - o(1))$.

Proof. We want to know the number of isomorphism classes of all $2^{\binom{n}{2}}$ graphs on n vertices. The isomorphism class of an asymmetric graph consists of all $n!$ permutations of its vertex set (and no less, because of the asymmetry). Together with Theorem 11.2, we get that there are $\frac{1}{n!}2^{\binom{n}{2}}(1 - o(1))$ nonisomorphic asymmetric graphs on n vertices. Again by Theorem 11.2, there are $2^{\binom{n}{2}}o(1)$ symmetric graphs. The isomorphism class of a symmetric graph consists of less than the $n!$ permutations of its vertex set, so there are between $\frac{1}{n!}2^{\binom{n}{2}}o(1)$ and $2^{\binom{n}{2}}o(1)$ nonisomorphic symmetric graphs on n vertices. But asymptotically, these amounts are both $2^{\binom{n}{2}}o(1)$. So the number of nonisomorphic graphs on n vertices is $\frac{1}{n!}2^{\binom{n}{2}}(1-o(1))+2^{\binom{n}{2}}o(1) = \frac{1}{n!}2^{\binom{n}{2}}(1-o(1))$. The second term is absorbed by the first one since $2^{\binom{n}{2}}$ increases much faster than $n!$. □

Lemma 11.5

Consider all tuples (Γ, C), where Γ is a graph with fixed vertex set $V = \{v_1, v_2, \dots, v_n\}$ and C is a GM-switching set of Γ of size 4, such that the following hold:

(i) The complement of C is asymmetric.

(ii) For every one of the three partitions of C into two pairs, there is a vertex outside C that has neighbours in one part but not in the other.

The number of such tuples is $\binom{n}{4}2^{\binom{n-1}{2}}(1 - o(1))$.

Proof. There are $\binom{n}{4}$ choices for the switching set. Fix a vertex $v \in C$ and choose the $2^{\binom{n-1}{2}}$ adjacencies between the $n-1$ other vertices.

We first prove that all adjacencies with v are fixed in a unique way. The edges between v and its neighbours in C are uniquely determined such that the induced subgraph on C is regular. Figure 11.1 illustrates this: if the adjacencies between the three other vertices are given, then there is a unique way to add edges from v such that the result is regular. For every vertex u outside C, we must add the edge uv if and only if u still has an odd number of neighbours in C.

Figure 11.1. Adjacencies with v are uniquely determined.

We now show that the proportion of *bad* choices of the adjacencies between the $n-1$ other vertices, i.e. those choices which harm one of the two given conditions, is $o(1)$. We know from Theorem 11.2 that the first condition is almost always fulfilled. For the second condition, consider an arbitrary partition of C into two pairs. There are 8^{n-4} possible choices for the adjacencies between the vertices outside C and the three vertices in $C \setminus \{v\}$. Out of these choices, there are 6^{n-4} for which the condition is harmed (for a fixed vertex outside C, of the 8 possible adjacencies with $C \setminus \{v\}$, there are only 2 ways in which that vertex has neighbours in one part but not in the other). So the proportion of bad choices is $\left(\frac{6}{8}\right)^{n-4} = o(1)$. Note that these conditions combined also hold for almost all constructed graphs because $(1-o(1)) \cdot (1-o(1)) = 1-o(1)$. □

Lemma 11.6

The number of triples (Γ, C, C') with Γ a graph with fixed vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and C and C' different GM-switching sets of Γ of size 4, is at most $\binom{n}{4}\left(\binom{n}{4}-1\right)2^{\binom{n-2}{2}}$.

Proof. There are $\binom{n}{4}$ choices for C and $\binom{n}{4}-1$ choices for C'. We separate two cases, depending on whether C and C' are disjoint.

If C and C' are disjoint, then fix two vertices $v \in C$ and $w \in C'$. We are free to choose the $2^{\binom{n-2}{2}}$ adjacencies between the $n-2$ other vertices, but then all adjacencies with v and w are fixed in a unique way like in the proof of Lemma 11.5. The only step where this could go wrong is when we determine the adjacency between v and w. So there are at most $2^{\binom{n-2}{2}}$ graphs in this case.

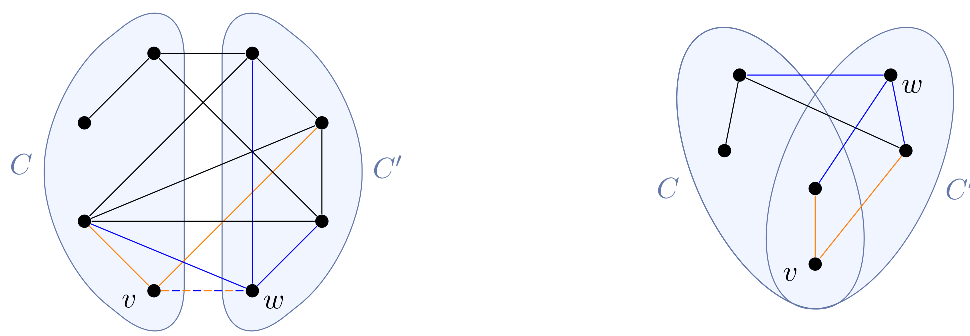


Figure 11.2. Adjacencies with v and w are uniquely determined.

If C and C' are not disjoint, then let v be a vertex in their intersection and $w \in C' \setminus C$. Like before, we can first choose the $2^{\binom{n-2}{2}}$ adjacencies between the $n-2$ other vertices, and then all adjacencies with v and w are fixed. To see this, first fix the adjacencies with v and then those with w. This could only go wrong when w ends up having an odd number of neighbours in C'. So there are again at most $2^{\binom{n-2}{2}}$ graphs with those switching sets. □

Theorem 11.7 ([42, Theorem 3])

The number of nonisomorphic graphs on n vertices that have a cospectral mate via GM-switching with respect to a 4-set is at least

$$n^3 g_{n-1}\left(\frac{1}{24} - o(1)\right)$$

where g_{n-1} denotes the number of nonisomorphic graphs on $n-1$ vertices.

Proof. Let n_i denote the number of graphs with fixed vertex set $V = \{v_1, v_2, \ldots, v_n\}$ that have exactly i different GM-switching sets of size 4. Let m_i denote the number of graphs with fixed vertex set $V = \{v_1, v_2, \ldots, v_n\}$ that have exactly i different GM-switching sets of size 4, all of which fulfil the extra conditions of Lemma 11.5. Clearly, $m_i \leq n_i$. Lemma 11.5 implies that

$$\binom{n}{4} 2^{\binom{n-1}{2}}(1 - o(1)) = \sum_{i=1}^{\infty} m_i \cdot i$$

and Lemma 11.6 implies that

$$\binom{n}{4}\left(\binom{n}{4} - 1\right) 2^{\binom{n-2}{2}} \geq \sum_{i=2}^{\infty} n_i \cdot i(i-1).$$

Combining these two equations yields the following lower bound on m_1:

$$\begin{aligned}
m_1 &= \binom{n}{4} 2^{\binom{n-1}{2}}(1 - o(1)) - \sum_{i=2}^{\infty} m_i \cdot i \\
&\geq \binom{n}{4} 2^{\binom{n-1}{2}}(1 - o(1)) + \sum_{i=2}^{\infty} \left(n_i \cdot i(i-1) - m_i \cdot i\right) - \binom{n}{4}\left(\binom{n}{4} - 1\right) 2^{\binom{n-2}{2}} \\
&\geq \binom{n}{4} 2^{\binom{n-1}{2}}\left((1 - o(1)) - \left(\binom{n}{4} - 1\right) 2^{2-n}\right) + \sum_{i=2}^{\infty} \left(n_i \cdot i(i-2)\right) \\
&\geq \binom{n}{4} 2^{\binom{n-1}{2}}\left((1 - o(1)) - \left(\binom{n}{4} - 1\right) 2^{2-n}\right) \\
&= \binom{n}{4} 2^{\binom{n-1}{2}}(1 - o(1))
\end{aligned}$$

where in the third step, we used that $m_i \leq n_i$ and $2^{\binom{n-2}{2} - \binom{n-1}{2}} = 2^{2-n}$, and in the fifth step that 2^n increases faster than any polynomial function in n. This inequality implies that the number of nonisomorphic graphs on n vertices that have a unique switching set of size 4 and fulfill the conditions of Lemma 11.5 is at least

$$\frac{1}{n!}\binom{n}{4} 2^{\binom{n-1}{2}}\left(1 - o(1)\right).$$

Using Corollary 11.4, we can rewrite this number as

$$\frac{(n-1)!}{n!}\binom{n}{4} g_{n-1}\left(1 - o(1)\right) = n^3 g_{n-1}\left(\frac{1}{24} - o(1)\right)$$

where g_{n-1} denotes the number of nonisomorphic graphs on $n-1$ vertices.

Finally, consider one of those graphs. We are left to prove that switching with respect to the unique switching set C of size 4 indeed gives a nonisomorphic graph. Suppose, for a contradiction, that the resulting graph is isomorphic. This isomorphism would then fix the switching set (because there is only one) and its complement. But the latter is asymmetric, and all adjacencies within the complement of C are retained, so the isomorphism must fix every vertex outside C. Because of the second condition, every vertex in C is mapped to the other part of C, for every partition of C into two pairs. But that is impossible. □

Since GM-switching also provides cospectral graphs with respect to the Laplacian matrix, the signless Laplacian matrix and the adjacency matrix of the complement [26], we also proved a lower bound for graphs that are NDS with respect to these matrices.

Recall from Chapter 2 that GM-switching with respect to a set of size 4 is the same as WQH-switching with respect to a set of size 4 $(2+2)$. So we could as well write “GM-switching” instead of “WQH-switching” in the statement of Theorem 11.7.

Part IV

Conclusion

Chapter 12

Conclusion and future work

In Chapter 4, we listed all current results on the cospectrality of generalized Johnson and Grassmann graphs. As we saw in the tables of Section 4.2, there are still many open questions, especially when the parameter k of these graphs is large.

We found two new infinite families of generalized Johnson and Grassmann graphs that are not determined by their spectrum, by applying both GM- and WQH-switching. More generally, we conclude that both switching techniques are useful for discovering cospectral mates. They have been proven successful in the past, and they still are.

In Part III, we gave new expressions for the diameter and girth of generalized Grassmann graphs. We also provided an elaborate proof of an asymptotic lower bound on the number of cospectral graphs by use of GM-switching.

12.1 Future work

We propose the following work as (directions for) future research.

(i) Many generalized Johnson and Grassmann are still not known to be determined by their spectrum. Therefore, determining whether these graphs have cospectral mates remains an open problem. The smallest open cases are currently $J_{\{1\}}(8,4)$, $J_{\{0,1\}}(8,4)$ and $K(9,3)$. One can consider different (switching) techniques as well (for instance the ones in [2]).

(ii) There exist other natural extensions of q-Kneser graphs, besides the generalized Grassmann graphs, see Remark 3.17. It makes sense to study the cospectrality of these graphs as well, since they are again very symmetrical.

(iii) There are many ways to optimize the code of Appendix C. At the moment, the algorithm still generates switching sets that are mutually isomorphic (i.e. sets that can be mapped onto one another by an automorphism of the graph). There exist many techniques for generating structures in an isomorph-free way, see [54]. Another trick that could optimize our code, is by using multiple GPUs (general purpose Graphical Processing Units) like in the program of [19], or the MPI (Message Passing Interface) library [61], which allows for parallel computation.

(iv) It might be interesting to look at the "automorphism group" of switching sets, i.e. the automorphisms of the graph that fix these sets on a global level (see also [54]). In particular, if the automorphism group of one of the sporadic cases in New result 4.10 would be one of the sporadic groups, then that would substantiate the fact that these switching sets really are sporadic.

(v) We expect the WQH-switching set in the proof of New result 4.10(iii) to be generalizable to switching sets for *all* generalized Grassmann graphs $J_q(n, n-2)$. More concrete, let us define the following set of vertices of $J_q(n, n-2)$.

Definition 12.1

Define the $(n-3)$-space $\pi := (0, 0, 0, \underbrace{*, *, \ldots, *}_{n-3})$ and the points

$$p_\alpha : = (1, \alpha, 0, 0, 0, \ldots, 0)$$
$$q_\alpha : = (1, 0, \alpha, 0, 0, \ldots, 0)$$

for all $\alpha \in \mathbb{F}_q$. Define the set $C := \{p_\alpha \pi \mid \alpha \in \mathbb{F}_q\} \cup \{q_\alpha \pi \mid \alpha \in \mathbb{F}_q\}$.

In other words, C consists of all $(n-2)$-spaces that are spanned by a fixed $(n-3)$-space π and a point on *exactly* one of two given intersecting lines ($(*, *, 0, 0, \ldots, 0)$ and $(*, 0, *, 0, 0, \ldots, 0)$) that span a plane that intersects the fixed $(n-3)$-space trivially. We have theoretical evidence that the set C is a WQH-switching set, and that the graph obtained by switching is not isomorphic to $J_q(n, n-2)$. In other words (and by using Lemma 3.16):

Work in progress 12.2

The Grassmann graphs $J_q(n, 2)$ are not determined by their spectrum if $n \geq 4$.

The details of the proof still need to be worked out. Since the graphs $J_q(n, 2)$ are strongly regular (see Theorem 1.8 and Theorem 3.14) and strongly regular graphs are characterized by their spectrum (Theorem 2.22), the obtained cospectral mates of $J_q(n, 2)$ will again be strongly regular, with the same parameters.

(vi) We could try to imitate the argumentation of Chapter 11 with switching sets of size 6 (two sets of size 3 in the case of WQH-switching). This again provides a bound on the number of graphs that are NDS, but we expect it to be weaker than the current one. If we would try to prove a variant of Lemma 11.5 for switching sets of size 6, then we cannot fix one vertex and choose all $2^{\binom{n-1}{2}}$ adjacencies between the other vertices such that they are extendable to a legitimate switching set. For example, the graph on 5 vertices and 1 edge cannot be extended to a graph of size 6 that is a switching set of a larger graph (this counterexample works for both GM- and WQH-switching). What we *can* do however, is fixing two vertices of the switching set and choosing all $2^{\binom{n-2}{2}}$ other adjacencies. Just like in Figure 11.1, we can extend all graphs of size 4 to a switching set of size 6, though not necessarily in a unique way. So the number of tuples (Γ, C) will be at least $\binom{n}{6} 2^{\binom{n-2}{2}}(1 - o(1))$. We expect the rest of the proof to be more or less the same, such that the bound is given by

$$n^4 g_{n-2} \left(\frac{1}{720} - o(1) \right)$$

where g_{n-2} denotes the number of nonisomorphic graphs on $n-2$ vertices. This weaker lower bound would then give some theoretical evidence that switching sets of size 4 are the most productive.

Appendix

Appendix A

Nederlandstalige samenvatting

Deze masterthesis bestaat uit vier delen. **Deel I** introduceert alle wiskundige structuren die aan bod zullen komen in de volgende delen. **Deel II** behandelt alle gekende resultaten over de cospectraliteit van veralgemeende Johnson- en Grassmanngrafen. In **Deel III** behandelen we twee hoofdstukken die gerelateerd zijn aan het vorige deel, maar eerder elk een onderwerp op zich vormen. We ronden deze thesis af met een besluit en enkele open problemen in **Deel IV**.

In **Hoofdstuk 1** beginnen we met een opfrissing van de belangrijkste begrippen binnen de grafentheorie, projectieve meetkunde en incidentiemeetkunde. In het vervolg werken we met vectoriële dimensies.

In **Hoofdstuk 2** maken we kennis met cospectrale grafen, een concept dat een cruciale rol speelt in de rest van ons verhaal:

Definitie A.1

Grafen zijn **cospectraal** als ze hetzelfde spectrum hebben. Een graaf is **bepaald door zijn spectrum** als elke graaf die er cospectraal mee is, er ook isomorf mee is.

Met "cospectraliteit" bedoelen we de eigenschap van grafen die zegt of ze al dan niet bepaald worden door hun spectrum. Hoewel veel structurele informatie bevat is in het spectrum van een graaf, zijn niet alle grafen bepaald door hun spectrum. Een belangrijke drijfveer voor dit onderzoek is het volgende vermoeden, geformuleerd door van Dam en Haemers in 2003.

Vermoeden A.2 ([26])

Bijna alle grafen zijn bepaald door hun spectrum.

Om meer inzicht te verwerven in het vermoeden, onderzoeken we de cospectraliteit van enkele families van grafen. Bewijzen dat een graaf bepaald is door zijn spectrum, is vaak moeilijk. Daarom richten we ons vooral op de constructie van cospectrale grafen.

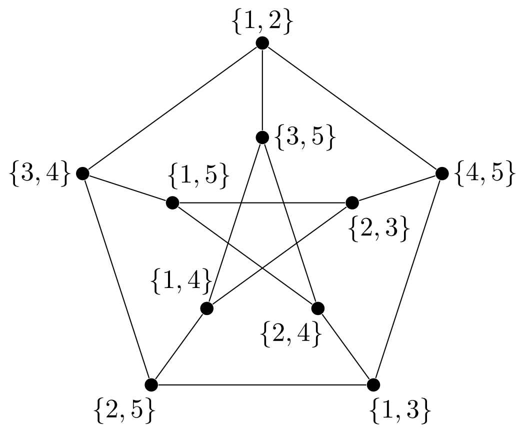
In het vervolg van **Hoofdstuk 2** bestuderen we cospectrale grafen voor zowel de adjacentiematrix als enkele andere matrices die men aan een graaf kan associëren. Daarna geven we een overzicht van de meest gekende technieken om cospectrale grafen te construeren.

Omdat veel van die technieken goed werken voor grafen die een zekere regulariteit vertonen, concentreren we ons op twee specifieke families met veel symmetrie: de veralgemeende Johnsongrafen en de veralgemeende Grassmanngrafen. We introduceren die grafen in **Hoofdstuk 3**. Ze worden als volgt gedefinieerd. Zij n en k natuurlijke getallen met $2 \leq k \leq n$.

Definitie A.3

Zij $S \subseteq \{0, 1, \ldots, k-1\}$. De **veralgemeende Johnsongraaf $J_S(n, k)$** heeft als toppen de deelverzamelingen van $\{1, 2, \ldots, n\}$ van grootte k, waarbij twee toppen adjacent zijn als de grootte van hun doorsnede (als verzamelingen) een element is van S.

$K(n, k) := J_{\{0\}}(n, k)$ noemen we de **Knesergraaf** en $J(n, k) := J_{\{k-1\}}(n, k)$ de **Johnsongraaf**.



Figuur A.1. De Petersengraaf $K(5, 2)$.

Zij q een priemmacht en $\mathbb{F}_q$ het eindig veld van orde q.

Definitie A.4

Zij $S \subseteq \{0, 1, \ldots, k-1\}$. De **veralgemeende Grassmanngraaf $J_{q,S}(n, k)$** heeft als toppen de deelruimten van $\mathbb{F}_q^n$ van dimensie k, waarbij twee toppen adjacent zijn als de dimensie van hun doorsnede (als deelruimten) een element is van S.

$K_q(n, k) := J_{q,\{0\}}(n, k)$ noemen we de **q-Knesergraaf** en $J_q(n, k) := J_{q,\{k-1\}}(n, k)$ de **q-Johnsongraaf** of **Grassmanngraaf**.

Het hoofddoel van deze thesis is om de cospectraliteit van veralgemeende Johnson- en Grassmanngrafen te bestuderen. In **Hoofdstuk 4** geven we een overzicht van alle eerder gekende resultaten, alsook enkele nieuwe. We presenteren de resultaten zowel chronologisch als in tabelvorm. Het hoofdstuk behandelt ook enkele concrete nieuwe gevallen van grafen waarvan we met de computer kunnen nagaan dat ze niet bepaald zijn door hun spectrum.

De overige hoofdstukken uit het tweede deel behandelen elk een apart resultaat over één of meerdere families van veralgemeende Johnson- en Grassmanngrafen. De eerste drie hoofdstukken gaan over eerder bewezen resultaten.

- In **Hoofdstuk 5** bewijzen we enkele straffe resultaten met behulp van punt-rechtemeetkunden. Meer bepaald kunnen we zo bekomen dat de (normale) Johnson- en Grassmanngrafen $J(n, k)$

en $J_q(n,k)$ niet bepaald zijn door hun spectrum zodra $3 \leq k \leq n-3$ [27]. Ook de grafen $J_q(2k+1,k)$ zijn op die manier niet bepaald zijn door hun spectrum [28]. We onderzoeken aan het einde van dit hoofdstuk hoe dit argument uitgebreid zou kunnen worden naar andere grafen.

- In **Hoofdstuk 6** bewijzen we dat $K(n,k)$ niet bepaald is door zijn spectrum als er een m bestaat zodanig dat $2 \leq m \leq k$ en $\binom{n-m}{k-m} = 2\binom{n-k-m}{k-m}$ [41]. We bewijzen ook dat de zogenaamde modulo 2 Knesergrafen $K_{\{0,2,4,\dots\}}(n,k)$ niet bepaald zijn door hun spectrum als $k \geq 3$ [41]. We sluiten het hoofdstuk af met enkele eigen observaties over hoe deze resultaten niet direct veralgemeenbaar zijn naar veralgemeende Grassmanngrafen (in het bijzonder de q-Knesergrafen).
- **Hoofdstuk 7** gaat over de cospectraliteit van twee specifieke families van Johnsongrafen. We tonen aan dat $J_{\{0,1,\dots,m\}}(3k-2m-1,k)$ niet bepaald is door zijn spectrum als $k \geq \max(m+2,3)$ en dat $J_{\{0,1,\dots,m\}}(n,2m+1)$ niet bepaald is door zijn spectrum als $m \geq 2$ en $n \geq 4m+2$ [19].

We beëindigen het tweede deel met twee nieuwe resultaten.

- In **Hoofdstuk 8** bewijzen we dat $J_{\{2\}}(n,4)$ niet bepaald is door zijn spectrum voor alle waarden van $n \geq 8$.
- In **Hoofdstuk 9** bewijzen we dat de q-Knesergrafen $K_q(n,k)$ niet bepaald zijn door hun spectrum als $q=2$.

Omdat veralgemeende Grassmanngrafen zo gestructureerd zijn, kunnen we expliciete uitdrukkingen vinden voor hun diameter en taille in functie van hun parameters. Dit gebeurt in **Hoofdstuk 10**. We baseren ons op reeds gekende uitdrukkingen voor de diameter en taille van veralgemeende Johnsongrafen [4, 62].

In **Hoofdstuk 11** bepalen we een ondergrens voor het aantal grafen die niet bepaald zijn door hun spectrum. Het bewijs uit [42] wordt daartoe in detail uitgewerkt.

Hoofdstuk 12 bevat het besluit van dit werk en enkele open problemen voor toekomstig onderzoek.

Appendix B

English summary

This master's thesis consists of four parts. **Part I** introduces all mathematical structures that are dealt with in the following parts. **Part II** is about all known results on the cospectrality of generalized Johnson and Grassmann graphs. In **Part III**, we cover two chapters that are related to the previous part, but are rather a topic on their own. We finish this thesis with a conclusion and some open problems in **Part IV**.

In **Chapter 1**, we refresh our memory with the most important notions in graph theory, projective geometry and incidence geometry. In the following, we will work with vectorial dimensions.

In **Chapter 2**, we learn about cospectral graphs, a concept that plays a crucial role in the rest of our story:

Definitie B.1

Graphs are **cospectral** if they have the same spectrum. A graph is **determined by its spectrum** if every graph that is cospectral with it, is also isomorphic to it.

With "cospectrality", we mean the property of graphs that tells whether or not they are determined by their spectrum. Although much structural information is contained in the spectrum of a graph, not all graphs are determined by their spectrum. An important driving force for this research is the following conjecture, formulated by van Dam en Haemers in 2003.

Vermoeden B.2 ([26])

Almost all graphs are determined by their spectrum.

To gain more insight into the conjecture, we investigate the cospectrality of some families of graphs. Proving that a graph is determined by its spectrum is often difficult. Therefore, we focus on the construction of cospectral graphs.

In the remainder of **Chapter 2**, we study cospectral graphs for both the adjacency matrix and several other matrices that can be associated with a graph. After that, we give an overview of the most well-known techniques for constructing cospectral graphs.

Since many of these techniques work well on graphs that show a certain regularity, we concentrate on two specific families with a lot of symmetry: the generalized Johnson graphs and the generalized

Grassmann graphs. We introduce those graphs in **Chapter 3**. They are defined as follows. Let n and k be natural numbers with $2 \leq k \leq n$.

Definitie B.3

Let $S \subseteq \{0, 1, \ldots, k-1\}$. The **generalized Johnson graph $J_S(n, k)$** has as vertices the subsets of $\{1, 2, \ldots, n\}$ of size k, where two vertices are adjacent if the size of their intersection (as sets) is an element of S.

$K(n, k) := J_{\{0\}}(n, k)$ is called the **Kneser graph** and $J(n, k) := J_{\{k-1\}}(n, k)$ the **Johnson graph**.

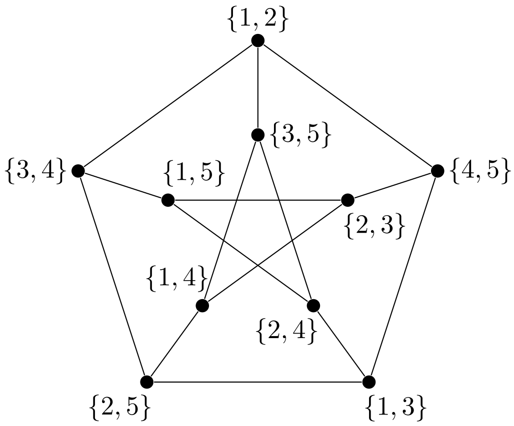


Figure B.1. The Petersen graph $K(5, 2)$.

Let q be a prime power and $\mathbb{F}_q$ the finite field of order q.

Definitie B.4

Let $S \subseteq \{0, 1, \ldots, k-1\}$. The **generalized Grassmann graph $J_{q,S}(n, k)$** has as vertices the subspaces of $\mathbb{F}_q^n$ of dimension k, where two vertices are adjacent if the dimension of their intersection (as subspaces) is an element of S.

$K_q(n, k) := J_{q,\{0\}}(n, k)$ is called the **q-Kneser graph** and $J_q(n, k) := J_{q,\{k-1\}}(n, k)$ the **q-Johnson graph** or **Grassmann graph**.

The main goal of this thesis is to study the cospectrality of generalized Johnson and Grassmann graphs. In **Chapter 4**, we give an overview of all previously known results, as well as some new ones. We present the results both chronologically as in tabular form. The chapter also treats some concrete new cases of graphs of which we can verify by computer that they are not determined by their spectrum.

The remaining chapters chapters from the second part each cover a seperate result about one or more families of generalized Johnson or Grassmann graphs. The first three of them deal with previously proved results.

- In **Chapter 5**, we prove some strong results using point-line geometries. More precisely, we can obtain in this way that the (normal) Johnson and Grassmann graphs $J(n, k)$ and $J_q(n, k)$ are not determined by their spectrum whenever $3 \leq k \leq n-3$ [27]. The graphs $J_q(2k+1,k)$ are not determined by their spectrum in this way either [28]. At the end of the chapter, we investigate how this argument could be extended to other graphs.

- In **Chapter 6**, we prove that $K(n,k)$ is not determined by its spectrum if there exists an m such that $2 \leq m \leq k$ and $\binom{n-m}{k-m} = 2\binom{n-k-m}{k-m}$ [41]. We also prove that the so-called modulo 2 Kneser graphs $K_{\{0,2,4,\dots\}}(n,k)$ are not determined by their spectrum if $k \geq 3$ [41]. We conclude the chapter with a few own observations on how these results are not directly generalizable to generalized Grassmann graphs (in particular the q-Kneser graphs).
- **Chapter 7** is about the cospectrality of two specific families of Johnson graphs. We show that $J_{\{0,1,\dots,m\}}(3k-2m-1,k)$ is not determined by its spectrum if $k \geq \max(m+2,3)$ and that $J_{\{0,1,\dots,m\}}(n,2m+1)$ is not determined by its spectrum if $m \geq 2$ and $n \geq 4m+2$ [19].

We end the second part with two new results.

- In **Chapter 8**, we prove that $J_{\{2\}}(n,4)$ is not determined by its spectrum for all values of $n \geq 8$.
- In **Chapter 9**, we prove that the q-Kneser graphs $K_q(n,k)$ are not determined by their spectrum if $q=2$.

Since the generalized Grassmann graphs are so structured, we can find explicit expressions for their diameter and girth in terms of their parameters. This happens in **Chapter 10**. We base ourselves on already known expressions for the diameter and girth of generalized Johnson graphs [4, 62].

In **Chapter 11**, we determine a lower bound on the number of graphs that are not determined by their spectrum. To that end, the proof from [42] is worked out in detail.

Chapter 12 contains the conclusion of this work and some open problems for future research.

Appendix C

Code

In this appendix, we give an overview of the code that was written for this thesis. We use the programming language `Python` [59].

The main purpose of our code is to exhaustively search for GM-and WQH-switching sets of a given size in a given generalized Johnson or Grassmann graph. More generally, it can be used to verify whether a given subset of vertices is a switching set in a given graph, and it provides an implementation of the generalized Johnson and Grassmann graphs.

The preamble looks as follows.

```
import itertools
from collections import Counter

# only needed for checking graph isomorphism when a switching set is found
from pynauty import *

# only needed for Grassmann graphs
import galois
import np
```

The code is divided into three parts.

1. The class `graph` represents any graph. It provides an implementation of the GM- and WQH-switching techniques from Definition 2.12 and Definition 2.14.
2. The class `Johnson` is a subclass of `graph` and implements the generalized Johnson graphs. Its main purpose is to search for all possible switching sets of a given size in these graphs.
3. The class `Grassmann` is a subclass of `graph` and implements the generalized Grassmann graphs. It is used for enumerating all possible switching sets of a given size in these graphs.

C.1 Switching techniques

```
class graph:
    def __init__(self, vertices, adjacency):
        self.vertices = vertices
        self.N = len(vertices)
        self.adjacent = adjacency
```

The adjacency dict of the graph will be labeled by integers, since otherwise it is not compatible with the `isomorphic` method below.

```
    # returns the adjacency dict, where vertices are labeled by 0,1,...,N
    def adjacency_dict(self):
        return {i: [j for j in range(self.N) if self.adjacent(self.vertices[i], self.vertices[j])] for i in range(self.N)}
```

We will frequently request the following method. It is a direct translation of Definition 2.12 into an algorithm.

```
    # checks whether C is a valid GM-switching set
    def is_gm_switching_set(self, C):
        # C must be regular
        valency = len([v for v in C if self.adjacent(C[0], v)])
        for v in C:
            if len([w for w in C if self.adjacent(v, w)]) != valency:
                return False
        # every vertex outside C must have 0,|C|/2 or |C| neighbours in C
        for v in self.vertices:
            if v not in C and len([w for w in C if self.adjacent(v, w)]) not in [0, len(C), len(C) / 2]:
                    return False
        return True
```

WQH-switching sets can either be given by two disjoint sets C_1 and C_2, or directly by their union C, without specifying how they are partitioned. We can, however, derive what this partition is when there exists a vertex outside C with an odd number of neighbours in C.

```
    # checks whether C = C1 U C2 is a valid WQH-switching set
    def is_wqh_switching_set(self, C):
        # first check if the partition C1 U C2 can already be derived
        for v in self.vertices:
            if v not in C:
                neighbours = [w for w in C if self.adjacent(v, w)]
                if len(neighbours) % 2 == 1:
                    if len(neighbours) == len(C) / 2:
                        return self.are_wqh_switching_sets(neighbours, [v for v in C if v not in neighbours])
                    return False
        for C1 in itertools.combinations(C, int(len(C) / 2)):
            # w.l.o.g.
            if C[0] in C1:
                if self.are_wqh_switching_sets(C1, [v for v in C if v not in C1]):
                    return True
        return False
```

```
    # checks whether (C1, C2) are valid WQH-switching sets
    def are_wqh_switching_sets(self, C1, C2):
        # "regularity"
        valency = len([v for v in C1 if self.adjacent(C1[0], v)]) - len([v for v in C2 if self.adjacent(C1[0], v)])
        for (Ci, Cj) in [(C1, C2), (C2, C1)]:
```

```
            for v in Ci:
                if len([w for w in Ci if self.adjacent(v, w)]) - len([w for w in Cj if self.adjacent(v, w)]) != valency:
                    return False
        # every vertex outside C1 U C2 has either |C1| neighbours in C1 and 0 in C2, 0 in C1 and |C2| in C2 or equally many in C1 and C2
        for v in self.vertices:
            if v not in C1 and v not in C2:
                e1 = len([w for w in C1 if self.adjacent(v, w)])
                e2 = len([w for w in C2 if self.adjacent(v, w)])
                if not (e1 == len(C1) and e2 == 0) and not (e1 == 0 and e2 == len(C2)) and not e1 == e2:
                    return False
        return True
```

While generating all possible GM-switching sets, the algorithm builds these sets recursively, which can take a lot of time. Luckily, the following method “prunes” a lot of these sets about halfway in the search tree. The idea is that vertices outside a GM-switching set C cannot have more than $\frac{1}{2}|C|$ neighbours in C and at the same time be nonadjacent with at least one vertex of C.

```
    # returns a vertex that must be contained in any GM_switching set of size "aim" that includes the given subgraph
    def gm_forced(self, aim, subgraph):
        for v in self.vertices:
            # the GM-switching property will never be satisfied if v is outside the subgraph and has strictly more than (aim)/2 neighbours and at least one nonadjacent vertex in the subgraph or strictly more than (aim)/2 nonadjacent vertices and at least one neighbour in the subgraph
            if v not in subgraph:
                e = len([w for w in subgraph if self.adjacent(v, w)])
                if e != 0 and e != len(subgraph) and (aim < 2 * e or aim < 2 * (len(subgraph) - e)):
                    return v
        return -1
```

We use the package pynauty for checking graph isomorphism. It is a Python/C extension module based on nauty [29, 55]. The following function determines whether the given GM-switching set produces a cospectral mate.

```
    # checks whether GM-switching about C returns a nonisomorphic graph
    def is_gm_successful(self, C):
        # relabel the vertices to numbers between 0 and N
        Cnew = [i for i in range(self.N) if self.vertices[i] in C]
        H = self.adjacency_dict()
        for v in H:
            if v not in Cnew and len(set(Cnew).intersection(H[v])) == len(Cnew) / 2:
                for w in Cnew:
                    if v in H[w]:
                        H[v].remove(w)
                        H[w].remove(v)
                    else:
                        H[v].append(w)
```

```
                    H[w].append(v)
        return not isomorphic(Graph(self.N, adjacency_dict=self.adjacency_dict()), Graph(self.N, adjacency_dict=H))
```

The same can be done for WQH-switching.

```
    # checks whether WQH-switching about C returns a nonisomorphic graph
    def is_wqh_successful(self, C):
        # relabel the vertices to numbers between 0 and N
        Cnew = [i for i in range(self.N) if self.vertices[i] in C]
        for C1 in itertools.combinations(C, int(len(C) / 2)):
            C2 = [v for v in C if v not in C1]
            if self.are_wqh_switching_sets(C1, C2):
                H = self.adjacency_dict()
                for v in H:
                    e1 = len([w for w in C1 if self.adjacent(self.vertices[v],w)])
                    e2 = len([w for w in C2 if self.adjacent(self.vertices[v],w)])
                    if (v not in Cnew) and ((e1 == len(C1) and e2 == 0) or (e1 == 0 and e2 == len(C2))):
                        for w in Cnew:
                            if v in H[w]:
                                H[v].remove(w)
                                H[w].remove(v)
                            else:
                                H[v].append(w)
                                H[w].append(v)
                if not isomorphic(Graph(self.N, adjacency_dict=self.adjacency_dict()), Graph(self.N, adjacency_dict=H)):
                    # print the partition to know what it is
                    print(list(C1), list(C2))
                    return True
        return False
```

C.2 Generalized Johnson graphs

Recall from Definition 3.8 that the vertices of the generalized Johnson graph $J_S(n,k)$ are the k-subsets of $\{1,2,\ldots,n\}$. We implement these sets as binary numbers, where the ith bit from the right is a 1 if and only if the corresponding set contains the element i. For example, the set $\{1,3,4,10\}$ is represented by the number $1000001101 = 525$. In this way, intersections and unions of sets translate to the bitwise logical operators AND (&) and OR (|). The vertices of the generalized Johnson graph $J(n,k)$ are then the integers below 2^n with exactly k ones. The number of ones of each of these integers is saved in the list `weight`. The command `x.bit_count()` is also efficient, but it requires `Python` 3.10.

```
class Johnson(graph):
    def __init__(self, S, n, k):
        self.n = n
        self.k = k
        self.weight = [bin(x).count("1") for x in range(2 ** n)]
        vertices = [x for x in range(2 ** n) if self.weight[x] == k]
        super().__init__(vertices, adjacency=lambda x, y: self.intersection_size(x, y) in S)
```

```
    def intersection_size(self, x, y):
        return self.weight[x & y]
```

The following method is our main tool for enumerating switching sets. It supports both GM- and WQH-switching.

```
    # returns all switching sets of J(S,n,k) of the required size
    def find_switching_sets(self, size, technique):
        self.switching_sets = []
        self.build_switching_sets(size, [], [], set([]), technique)
        print(str(len(self.switching_sets)) + " switching sets of size " + str(size) + " found.")
        if self.switching_sets:
            successful = False
            for C in self.switching_sets:
                if (technique == "GM" and self.is_gm_successful(C)) or (technique == "WQH" and self.is_wqh_successful(C)):
                    successful = True
                    print("The switching set " + str(C) + " produces a cospectral mate.")
            if not successful:
                print("Too bad. The switching sets produce isomorphic graphs.")
```

The very core of our algorithm is the method `build_switching_sets(aim, subgraph, invariants, forced_vertices, technique)`. It builds up all possible switching sets in a recursive way, starting from `subgraph` until a set of size `aim` is reached. If the `technique` is `"GM"`, then the elements of `forced_vertices` are vertices that must eventually be contained in the GM-switching set (see the method `gm_forced(aim, subgraph)` above). However, they are not directly added to the `subgraph` because we **add the vertices in a certain order**. This prevents us from generating the same switching set too many times.

The order that we are considering, is the following. For each vertex in the `subgraph`, we define its **invariant** as the sum of the squares of the intersection sizes of the corresponding k-sets with all other vertices in the `subgraph`. We only continue the algorithm if the new vertex has the largest unique invariant, or just the largest if all invariants occur at least twice. This is a valid strategy, since we can work backwards from any given switching set and delete each time the vertex with the highest (unique) invariant. For more information on this strategy, see [54]. Note that the number representing the set would be a useful invariant as well. However, we observed that the previously defined one performs better.

The function `build_vertices(current_elements, current_position)` does a similar thing, but on the level of vertices (k-sets). It recursively adds elements (i.e. it adds ones in a binary number) in order to build up a k-set. We again do this in a certain order, which is just from right to left in the binary representation. If there are "isomorphic" elements that belong to exactly the same k-sets in the current `subgraph`, and we decide not to add the lowest of these elements, then the others are not added either. This is a valid strategy, since we can always swap such isomorphic elements by an automorphism of the graph, while keeping the `subgraph` invariant. In other words, we only have to consider *initial segments* among sets of elements that behave the same, much like the very first vertex can be chosen as the set $\{1, 2, \ldots, k\}$ without loss of generality.

```
    # appends to "switching_sets" all switching sets of the required size "aim" that contain the given subgraph, where the list "invariants"
```

```
    contains their respective invariants and "forced_vertices" the k-sets that must be contained in the switching set at some point
    def build_switching_sets(self, aim, subgraph, invariants, forced_vertices, technique):
        # union of the subgraph and the forced vertices
        full_subgraph = forced_vertices.union(subgraph)
        # too many vertices
        if len(full_subgraph) > aim:
            return
        # if we have a switching set of size aim
        if len(subgraph) == aim:
            if (technique == "GM" and self.is_gm_switching_set(subgraph)) or (technique == "WQH" and self.is_wqh_switching_set(subgraph)):
                self.switching_sets.append(subgraph[:])
            return
        # if we are forced to add a vertex to the subgraph (only possible for GM-switching)
        if technique == "GM":
            x = self.gm_forced(aim, full_subgraph)
            if x != -1:
                forced_vertices.add(x)
                self.build_switching_sets(aim, subgraph, invariants, forced_vertices, technique)
                forced_vertices.remove(x)
                return
        # add a vertex to the subgraph if the order is respected
        def add_vertex(x):
            if x not in subgraph:
                # idea: w.l.o.g. only add x if its invariant is largest among all invariants in subgraph U {x}, where an invariant is the sum of the squares of the intersection sizes with all other elements of subgraph U {x}
                new_invariant = 0
                for i in range(len(subgraph)):
                    to_add = self.intersection_size(subgraph[i], x) ** 2
                    invariants[i] += to_add
                    new_invariant += to_add
                invariants.append(new_invariant)
                frequencies = Counter(invariants)
                # if new_invariant is the highest unique invariant, or just highest if there are no unique invariants
                if (1 in frequencies.values() and frequencies[new_invariant] == 1 and new_invariant == max([y for y in invariants if frequencies[y] == 1])) or (1 not in frequencies.values() and new_invariant >= max(invariants, default=0)):
                    subgraph.append(x)
                    if x in forced_vertices:
                        forced_vertices.remove(x)
                        self.build_switching_sets(aim, subgraph, invariants, forced_vertices, technique)
                        forced_vertices.add(x)
                    else:
                        self.build_switching_sets(aim, subgraph, invariants, forced_vertices, technique)
                    subgraph.pop()
                invariants.pop()
                for i in range(len(subgraph)):
                    invariants[i] -= self.intersection_size(subgraph[i], x)
```

```
                    ** 2
        # calculate the support of C (all elements that are in some k-set of C)
        support = 0
        for x in subgraph:
            support |= x
        # applies the function "add_vertex" to every possible vertex, up to certain isomorphism, by recursively adding elements to enumerate all k-sets
        # we go from right to left in binary representation (i.e. in increasing order) and keep track of the currently added elements (a binary number with weight <= k)
        # "current position" is a binary number with one 1, like 000100000 for the 5th position
        def build_vertices(current_elements, current_position):
            if self.weight[current_elements] == self.k:
                # current_elements is now a k-set
                add_vertex(current_elements)
                return
            if current_position & support == 0:
                # no other k-set in the subgraph contains elements from here on, so we only consider the smallest case since all other cases will be isomorphic
                for _ in range(self.k - self.weight[current_elements]):
                    if current_position >= 2 ** self.n:
                        # we reached the nth element and cannot add any more elements
                        return
                    # add a one in this position
                    current_elements += current_position
                    # move one bit to the left
                    current_position *= 2
                build_vertices(current_elements, 0)
                return
            # either we add the next element to current_elements
            build_vertices(current_elements + current_position, current_position * 2)
            # or we skip some elements until the next "nonisomorphic" element
            skip = True
            while skip:
                for x in subgraph:
                    # check whether containment in x is the same for the element on this position as it is for the next one
                    if not x & current_position == (x >> 1) & current_position:
                        # the subsequent elements have a different role
                        skip = False
                        break
                # move one bit to the left
                current_position *= 2
            build_vertices(current_elements, current_position)
        # start the recursion
        build_vertices(0, 1)
```

C.3 Generalized Grassmann graphs

We use the package `galois` for doing calculations in finite fields [45]. We represent the k-subspaces of $\mathbb{F}_q^n$ by $(k \times n)$-matrices where each row is a basis vector. More specifically, we implement them as matrices in *reversed reduced row echelon form*. For example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix}$$

represents a 3-subspace of a 5-space.

```python
class Grassmann(graph):
    def __init__(self, q, S, n, k):
        self.n = n
        self.k = k
        self.GF = galois.GF(q)
        self.k_spaces = []
        # add all k-space recursively, where each k-space is given by a matrix of basis vectors in reversed reduced row echelon form
        # given the first rows and their pivots, add a basis vector
        def add_spaces(basis, pivots, lastpivot):
            if len(basis) == k:
                self.k_spaces.append([v[:] for v in basis])
                return
            for newpivot in range(lastpivot + 1, self.n - self.k + len(basis) + 1):
                new_vector = [0] * self.n
                new_vector[newpivot] = 1
                pivots[newpivot] = True
                while True:
                    basis.append(new_vector)
                    add_spaces(basis, pivots, newpivot)
                    basis.pop()
                    increment = False
                    for i in range(newpivot):
                        if not pivots[i]:
                            new_vector[i] += 1
                            if new_vector[i] < q:
                                increment = True
                                break
                            new_vector[i] = 0
                    if not increment:
                        break
                pivots[newpivot] = False
        add_spaces([], [False] * n, -1)
        self.intersections = [[-1] * len(self.k_spaces) for _ in range(len(self.k_spaces))]
        # vertices are indices, the 'real' vertices are in "k-spaces"
        super().__init__([x for x in range(len(self.k_spaces))], adjacency=lambda x, y: self.intersection_dim(x, y) in S)
```

Determining the intersection dimension of two such spaces comes down to calculating the rank of a matrix. Since this is rather intensive, we keep track of already computed intersection dimensions in `intersections`.

```python
    def intersection_dim(self, x, y):
        if self.intersections[x][y] != -1:
            return self.intersections[x][y]
        # Grassmann identity
        dimension = 2 * self.k - np.linalg.matrix_rank(self.GF(self.k_spaces[x] + self.k_spaces[y]))
        self.intersections[x][y] = dimension
        self.intersections[y][x] = dimension
        return dimension
```

The following piece of code is virtually the same as for the `Johnson` class.

```python
    def find_switching_sets(self, size, technique):
        self.switching_sets = []
        self.build_switching_sets(size, [], [], set([]), technique)
        print(str(len(self.switching_sets)) + " switching sets of size " + str(size) + " found.")
        if self.switching_sets:
            successful = False
            for C in self.switching_sets:
                if (technique == "GM" and self.is_gm_successful(C)) or (
                        technique == "WQH" and self.is_wqh_successful(C)):
                    successful = True
                    print("The switching set " + str(C) + " produces a cospectral mate.")
            if not successful:
                print("Too bad. The switching sets produce isomorphic graphs.")
```

Building up switching sets is very similar to the method of `Johnson` with the same name. Again, we define an **invariant**, which is now the sum of the squares of the intersection *dimensions* with all other vertices in `subgraph`, and again, we only add those vertices with the largest (unique) invariant. The only fundamental difference with the generalized Johnson graphs, is that we cannot build up k-sets in an easy way. Instead, we just add *all* possible vertices. However, in the first and second step, restrictions can be made.

```python
    # appends to "switching sets" all switching sets of the required size "aim" that contain the given subgraph, where the list "invariants" contains their respective invariants and "forced_vertices" the k-sets that must be contained in the switching set at some point
    def build_switching_sets(self, aim, subgraph, invariants, forced_vertices, technique):
        # union of the subgraph and the forced vertices
        full_subgraph = forced_vertices.union(subgraph)
        # too many vertices
        if len(full_subgraph) > aim:
            return
        # if we have a switching set of size aim
        if len(subgraph) == aim:
            if (technique == "GM" and self.is_gm_switching_set(subgraph)) or (technique == "WQH" and self.is_wqh_switching_set(subgraph)):
                self.switching_sets.append(subgraph[:])
            return
        # if we are forced to add a vertex to the subgraph (only possible for
```

```
            GM-switching)
        if technique == "GM":
            x = self.gm_forced(aim, full_subgraph)
            if x != -1:
                forced_vertices.add(x)
                self.build_switching_sets(aim, subgraph, invariants, forced_vertices, technique)
                forced_vertices.remove(x)
                return
        # add a vertex to the subgraph if the order is respected
        def add_vertex(x):
            if x not in subgraph:
                # idea: w.l.o.g. only add x if its invariant is largest among all invariants in subgraph U {x}, where an invariant is the sum of the squares of the intersection sizes with all other elements of subgraph U {x}
                new_invariant = 0
                for i in range(len(subgraph)):
                    to_add = self.intersection_dim(subgraph[i], x) ** 2
                    invariants[i] += to_add
                    new_invariant += to_add
                invariants.append(new_invariant)
                frequencies = Counter(invariants)
                # if new_invariant is the highest unique invariant, or just highest if there are no unique invariants
                if (1 in frequencies.values() and frequencies[new_invariant] == 1 and new_invariant == max([y for y in invariants if frequencies[y] == 1])) or (1 not in frequencies.values() and new_invariant >= max(invariants, default=0)):
                    subgraph.append(x)
                    if x in forced_vertices:
                        forced_vertices.remove(x)
                        self.build_switching_sets(aim, subgraph, invariants, forced_vertices, technique)
                        forced_vertices.add(x)
                    else:
                        self.build_switching_sets(aim, subgraph, invariants, forced_vertices, technique)
                    subgraph.pop()
                invariants.pop()
                for i in range(len(subgraph)):
                    invariants[i] -= self.intersection_dim(subgraph[i], x) ** 2
        # applies the function "add_vertex" to every possible vertex
        if not subgraph:
            add_vertex(0)
            return
        if len(subgraph) == 1:
            for d in range(self.k):
                for v in self.vertices:
                    if self.intersection_dim(v, subgraph[0]) == d:
                        add_vertex(v)
                        break
            return
        for vertex in self.vertices:
            add_vertex(vertex)
```

C.4 Sporadic cospectral mates

Finally, we provide commands for finding the cospectral mates of the three sporadic graphs of New result 4.10. Note that many of the switching sets are isomorphic (or even the same).

```
G = Johnson(S={1}, n=11, k=4)
G.find_switching_sets(size=6, technique="WQH")
```

```
4 switching sets of size 6 found.
[15, 232, 1800] [240, 23, 1808]
The switching set [15, 240, 23, 232, 1800, 1808] produces a cospectral mate.
[15, 232, 1800] [240, 23, 1808]
The switching set [15, 240, 23, 232, 1808, 1800] produces a cospectral mate.
[15, 113, 1793] [240, 142, 1920]
The switching set [15, 240, 113, 142, 1793, 1920] produces a cospectral mate.
[15, 113, 1793] [240, 142, 1920]
The switching set [15, 240, 113, 142, 1920, 1793] produces a cospectral mate.
```

```
for x in [15, 232, 1800, 240, 23, 1808]:
    print(bin(x)[2:].zfill(11))
```

```
00000001111
00011101000
11100001000
00011110000
00000010111
11100010000
```

```
G = Johnson(S={2,4}, n=10, k=5)
G.find_switching_sets(size=10, technique="GM")
```

```
149916 switching sets of size 10 found.
The switching set [31, 47, 451, 707, 817, 818, 844, 908, 244, 248] produces a
    cospectral mate.
The switching set [31, 47, 451, 707, 817, 818, 244, 248, 844, 908] produces a
    cospectral mate.
The switching set [31, 47, 451, 707, 844, 908, 817, 818, 244, 248] produces a
    cospectral mate.
The switching set [31, 47, 451, 707, 844, 908, 244, 248, 817, 818] produces a
    cospectral mate.
The switching set [31, 47, 451, 707, 244, 248, 817, 818, 844, 908] produces a
    cospectral mate.
The switching set [31, 47, 451, 707, 244, 248, 844, 908, 817, 818] produces a
    cospectral mate.
The switching set [31, 47, 241, 242, 835, 899, 460, 716, 820, 824] produces a
    cospectral mate.
The switching set [31, 47, 241, 242, 835, 899, 820, 824, 460, 716] produces a
    cospectral mate.
The switching set [31, 47, 241, 242, 460, 716, 835, 899, 820, 824] produces a
    cospectral mate.
The switching set [31, 47, 241, 242, 460, 716, 820, 824, 835, 899] produces a
    cospectral mate.
The switching set [31, 47, 241, 242, 820, 824, 835, 899, 460, 716] produces a
    cospectral mate.
The switching set [31, 47, 241, 242, 820, 824, 460, 716, 835, 899] produces a
    cospectral mate.
```

```
for x in [31, 47, 451, 707, 817, 818, 844, 908, 244, 248]:
    print(bin(x)[2:].zfill(10))
```

```
0000011111
0000101111
0111000011
1011000011
1100110001
1100110010
1101001100
1110001100
0011110100
0011111000
```

```
G = Grassmann(q=3, S={0}, n=4, k=2)
G.find_switching_sets(size=6, technique="WQH")
```

```
936 switching sets of size 6 found.
[0, 1, 2] [4, 9, 11]
The switching set [0, 1, 2, 4, 9, 11] produces a cospectral mate.
[0, 1, 2] [4, 11, 9]
The switching set [0, 1, 2, 4, 11, 9] produces a cospectral mate.
[0, 1, 2] [5, 7, 12]
The switching set [0, 1, 2, 5, 7, 12] produces a cospectral mate.
[0, 1, 2] [5, 12, 7]
The switching set [0, 1, 2, 5, 12, 7] produces a cospectral mate.
[0, 1, 2] [6, 8, 10]
The switching set [0, 1, 2, 6, 8, 10] produces a cospectral mate.
.
.
.
[0, 1, 2] [128, 129, 127]
The switching set [0, 1, 128, 129, 127, 2] produces a cospectral mate.
[0, 1, 2] [129, 127, 128]
The switching set [0, 1, 129, 127, 2, 128] produces a cospectral mate.
[0, 1, 2] [129, 127, 128]
The switching set [0, 1, 129, 127, 128, 2] produces a cospectral mate.
[0, 1, 2] [129, 128, 127]
The switching set [0, 1, 129, 128, 2, 127] produces a cospectral mate.
[0, 1, 2] [129, 128, 127]
The switching set [0, 1, 129, 128, 127, 2] produces a cospectral mate.
```

```
for x in [0, 1, 2, 4, 9, 11]:
    print(G.k_spaces[x])
```

```
[[1, 0, 0, 0], [0, 1, 0, 0]]
[[1, 0, 0, 0], [0, 0, 1, 0]]
[[1, 0, 0, 0], [0, 1, 1, 0]]
[[1, 0, 0, 0], [0, 0, 0, 1]]
[[1, 0, 0, 0], [0, 2, 1, 1]]
[[1, 0, 0, 0], [0, 1, 2, 1]]
```

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