Multiplication of distributions

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Contents

Preface

The theory of distributions has shown over time its importance and has numerous applications in science and engineering. Specifically the fields of partial differential equations, quantum mechanics and signal processing benifit heavily from distribution theory. Core concepts like the distributional derivative and the Fourier transform for tempered distributions make it a very powerful theory. One of the major weaknesses of distribution theory however is the lack of a universal multplication of distributions. A lot of proposals to define a multiplication of distributions have been presented in the literature, some results even begin mutually contradictory. Over time the most most comprehensive approaches were further developed. In the first part of this thesis we study some of these well-behaved multiplications of distributions. We follow the first chapters of the book 'Multiplication of distributions and applications to partial differential equations' by Oberguggenberger [30]. Another way to multiply distributions is to extend the space of distributions to a larger space of generalised functions. The most succesful extension are the Colombeau algebras [10]. The product of Colombeau generalised functions is well-defined and the space of distributions can be embedded into the Colombeau algebras. Colombeau algebras have seen strong applications to partial differential equations with singular coefficients. The recently introduced concept of very weak solutions which orginated from the paper [19] by Garetto and Ruzhansky, is a new way to solve PDEs with singular coefficients. We present the very weak solution concept to the reader and in the second part of this thesis we apply the very weak solution concept to Euler-Bernoulli beam equation with discontinuous cross-section and singular coefficients. We also numerically investigate the solutions of the Euler-Bernoulli equation.

This masters thesis consists of three chapters. The first introductory chapter discusses standard concepts and useful results in distribution theory, functional analysis, partial differential equations and numerical analysis. This allows us to discuss the multplication problem and the very weak solution in a more straightforward manner.

Chapter two discusses the multiplication problem of distributions. It is known that some products, however defined, cannot be a distribution. The main focus of the chapter is on intrinsic multiplication for which the product is again distribution. Since some products are impossible, we can only define the multiplication partially. The more general the multiplication, the weaker the continuity properties are. The Schwartz product defines the product of a distribution and a smooth function. Through localisation it is extended to a product of distributions with disjoint singular support. Next we discuss the duality method. The duality method generalises the Schwartz product. If a subspace of distributios X is normal, then we obtain a product for distributions in the multiplier spaces X_{loc} and X'_{loc}. The Fourier product is based on the $\mathcal{S}'$-convolution and exists whenever the $\mathcal{S}'$-convolution of the Fourier transforms of the factors exists. Lastly we discuss the strict product and the model product. These products are most general and are defined through regularisation of the factors. The regularisations are achieved by convolution with strict and model delta nets respectively. For each of the products we prove properties and discuss examples. Now we discuss extrinsic multplication. Schwartz's famous impossibility result [33] says that the space of distributions cannot be extended to an associative differential algebra for which multiplication of continuous functions coincides with the pointwise multiplication. We shortly discuss the special Colombeau algebras for which pointwise multiplication coincides for smooth functions. Finally we explain the very weak solution concept through a simple example.

The third chapter concerns a very weak solution of the Euler-Bernoulli equation with distrubutional coefficients and numerical analysis of the beam solutions. The Euler-Bernoulli equation is a partial differential equation that describes the bending of a beam under several forces. It is commonly used in engineering to verify the stability of a beam as part of a construction. We consider the dynamic Euler-Bernoulli equation, which describes the movement of the beam over time and we will consider only vertical and axial forces. First we describe the physical modelling of the beam. Then we present a weak solution to the Euler-Bernoulli equation with L^∞ coefficients following [21]. This will be used to solve the regularised equation in our very weak solution method. We continue by defining a very weak solution to the Euler-Bernoulli equation with distributional coefficients. We prove existence and uniqueness theorems for the very weak solution. Lastly we use numerical analysis to investigate the solutions of the Euler-Bernoulli equation with distributional forces.

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Chapter 1

Mathematical preliminaries

In this chapter we introduce and define well-known concepts in functional analysis, distributin theory, partial differential equations and numerical analysis. Our goal is to cover the standard theory of these subjects to the extent we need them. We will also include lesser-known theorems we need later in some proofs and examples. This chapter is mostly based on graduate-level textbooks about distribution theory [18, 22], functional analysis [35], partial differential equations [11] and the finite element method [9, 25].

1.1 Foundational definitions and notation

In the framework of functional analysis functions and distributions are described as part of topological vector spaces (TVS).

Definition 1.1.1 (Vectorspace, Topological vectorspace)**.** Let $\mathbb{K}$ be a field. A vector space X over $\mathbb{K}$ is a system of three objects (X, A, M) consisting of a set X and of two mappings:

$$A : X \times X \to X, (x, y) \mapsto x + y,$$
$$M : \mathbb{K} \times X \to X, (\lambda, x) \mapsto \lambda x.$$

The map A called vector addition and is associative, commutative. There is a neutral element 0 for which $A(0, x) = 0$. For every $x \in X$ there is a negative element $-x$ for which $A(x, -x) = 0$. The map M is called scalar multiplication and satisfies:

$$\begin{aligned} \lambda(\mu x) &= \mu(\lambda x), \\ (\lambda + \mu)x &= \lambda x + \mu x, \\ \lambda(x + y) &= \lambda x + \lambda y, \\ 1x &= x, \\ 0x &= 0. \end{aligned}$$

A topological vectorspace X is a topological space and a vector space such that the maps A and M are continuous maps with respect the the topology of X.

Examples of topological vector spaces are normed vector spaces, Banach spaces and Fréchet spaces. Spaces of continuous linear functionals on a TVS can be endowed with topology to form a TVS structure as well.

We choose to characterize the topologies of function spaces using (semi-)norms and convergence of nets. We begin with normed spaces since they are considerably simpler.

Definition 1.1.2.

- Let X be a vector space. A norm on X is a map $|| \cdot || : X \to \mathbb{R}$ such that

$$||\lambda x|| = |\lambda| ||x||, \tag{1.1}$$
$$||x + y|| \leq ||x|| + ||y||, \tag{1.2}$$
$$||x|| = 0 \iff x = 0, \tag{1.3}$$

  for any $x, y \in X$.

- A seminorm $|\cdot|$ is a map $X \to \mathbb{R}$ which is satisfies (1.1) and (1.2).
- A TVS X is a normed space if there exists a norm $||\cdot||_X$ on X such that any sequence $\{v_n\}_{n\in\mathbb{N}} \subset X$ converges to $v \in X$ in the topology of X if and only if

$$||v_n - v||_X \to 0, \quad \text{as } n \to \infty.$$

- Let X be a normed space. A sequence $\{v_n\}_{n\in\mathbb{N}})$ in X is a Cauchy sequence if and only if

$$\forall\varepsilon,\, \exists N \in \mathbb{N},\, \forall m, n > N : ||v_n - v_m||_X \leq \varepsilon.$$

- A normed space is complete if every Cauchy sequence converges.
- A Banach space is a complete, normed vector space.

Example 1.1.3. Let $C([a,b])$ be the vectorspace of continuous functions $[a,b] \to \mathbb{C}$. It is a normed vectorspace with respect to the norm

$$||f||_{C([a,b])} = \max_{x\in[a,b]} |f(x)|.$$

△

Definition 1.1.4 (Upwards directed set)**.** Let I be a set and $\leq$ a pre-order on I. We call I upwards directed set if every pair of elements a and b of I have an upper bound c in I.

A pre-order is a relation that is transitive and reflexive, but not necessarily anti-symmetric. In that case we have a partial order.

Example 1.1.5. a) Simple examples are ordered sets like the natural numbers $\mathbb{N}$, the real numbers $\mathbb{R}$ and intervals (a,b) in $\mathbb{R}$.

b) Essential for us are directed sets of neighborhoods. Let X be a topological space and $x \in X$. Let $\mathcal{N}_x$ be the collection of all neighborhoods of x. This is the set of all open sets U which contain x. The intersection W of two neighborhoods of x, U and V is again a neighborhood of x. This means that V and W have the upper bound $W = U \cap V$ with respect to the reverse inclusion $\supseteq$. We conclude that $\mathcal{N}_x$ is a directed set with pre-order $\supseteq$.

△

Definition 1.1.6 (Net)**.** Let X be a set and I an upward directed set. A net in X is a map $I \to X$. We write $\{x_i\}_{i\in I}$. When the index set I is a connected subset of $\mathbb{R}$, then a net $\{x_\lambda\}_{\lambda\in I}$ is a parametrization.

Definition 1.1.7 (Limits of nets)**.** Let X be a topological space and $\mathcal{N}_x$ the set of neighborhoods of $x \in X$. A net $\{x_\lambda\}_{\lambda\in I}$ converges to x if

$$\forall U \in \mathcal{N}_x,\, \exists\lambda_0 \in I,\, \forall\lambda \geq \lambda_0 : x_\lambda \in U.$$

We write $x = \lim_{\lambda\in I} x_i$.

Example 1.1.8 (complex-valued limits)**.** Let $\{a_\lambda\}_{\lambda\in I} \subset \mathbb{C}$ be a net of complex numbers. The number $a \in \mathbb{C}$ is the limit of the net $\{a_\lambda\}_{\lambda\in I}$ if

$$\forall\varepsilon > 0,\, \exists\lambda_0 \in I,\, \forall\lambda \geq \lambda_0 : |a - a_\varepsilon| \leq \varepsilon.$$

△

We will also use nets for the index set $I = (0,1]$ with the reverse order $\geq$. This means we consider small ε as large in I and we have the limit $\lim_{\varepsilon\in I} x_\varepsilon = \lim_{\varepsilon\to 0} x_\varepsilon$.

Nets suffice to characterise continuity of mappings.

Proposition 1.1.9. *Let X, Y be topological spaces. A map $f : X \to Y$ is continuous if for very net $\{x_\lambda\}_{\lambda\in I}$ converging to some x in X, the net of images $\{f(x_\lambda)\}_{\lambda\in I}$ converges to $f(x)$ in Y.*

Instead of the usual topological approach, we will use nets, seminorms and norms to define the topologies of function spaces.

Definition 1.1.10 (Fréchet space)**.** A Fréchet space is a locally convex, complete and metrizable TVS.

The topology of a Fréchet space is characterized by countably many seminorms.

Theorem 1.1.11. *Let X be Fréchet space, then there exists a countable set of seminorms $\{|\cdot|_n, n \in \mathbb{N}\}$ such that any net $\{x_\lambda\}_{\lambda \in I}$ converges to $x \in X$ if and only if*

$$|x_\lambda - x|_n \to 0, \quad \text{for every } n \in \mathbb{N}.$$

This characterisation allows us to treat Fréchet spaces by their seminorms.

1.2 Function spaces

One of the most important classes of TVS are the function spaces. These are mappings $\mathbb{R}^n \to \mathbb{C}$ with additional properties. In all of the definitions below we will consider the domain $\mathbb{R}^n$. Similar definitions can be given on an open set $\Omega \subseteq \mathbb{R}^n$. First we introduce some notation.

Definition 1.2.1 (Multi-index)**.** A multi-index α is a tuple $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. The length of the multi-index α is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We write x^α for the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Let φ be a function $\mathbb{R}^n \to \mathbb{C}$. We denote the α-th partial derivative of φ by

$$\partial^\alpha \varphi = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} \varphi,$$

if the righthandside exists.

When using multi-index notation for derivatives, we should make sure that the result is independent of the order of derivation.

Definition 1.2.2 (L^p-space)**.** For $1 \leq p < \infty$ the p-norms are

$$||f||_p^p = \int_{\mathbb{R}^n} |f|^p,$$

and for $p = \infty$

$$||f||_\infty = \text{essup}_{x \in \mathbb{R}^n} |f| = \inf\{C \geq 0 : |f| \leq C, \text{ almost everywhere}\}.$$

The space L^p is the normed space of (equivalence classes) of functions with bounded p-norm.

Theorem 1.2.3. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $1 \leq p \leq \infty$ and $f \in L^p(\Omega)$, then*

$$f \in L^q(\Omega), \quad \text{for } p \leq q \leq \infty.$$

Theorem 1.2.4 (Hölder's inequality)**.** *Let $1 \leq p, q, r \leq \infty$ satisfy $\dfrac{1}{p} + \dfrac{1}{q} = \dfrac{1}{r}$. For every $f \in L^p$ and $g \in L^q$ we have*

$$||fg||_{L^r} \leq |f||_{L^p} ||g||_{L^q}. \tag{1.4}$$

Definition 1.2.5 (Smooth functions)**.** Define the seminorms

$$|\varphi|_{j,K} := \sup_{\substack{x \in K, \\ \alpha \in \mathbb{N}^n, |\alpha| \leq j}} |\partial^\alpha \varphi(x)|, \quad j \in \mathbb{N}, K \subset \mathbb{R}^n \text{ compact}. \tag{1.5}$$

We consider the space of infinitely differentiable functions $C^\infty(\mathbb{R}^n)$ as a Fréchet space with the topology induced by the seminorms (1.5). We will call it the space of smooth functions and use the alternative notation $\mathcal{E}(\mathbb{R}^n)$.

We elaborate a bit on the convergence in $\mathcal{E}(\mathbb{R}^n)$. By the definition a net of smooth functions $\{\varphi_\lambda\}_{\lambda \in I}$ converges in $\mathcal{E}(\mathbb{R}^n)$ to the function $\varphi \in \mathcal{E}(\mathbb{R}^n)$ when

$$|\varphi_\lambda - \varphi|_{j,K} \to 0,$$

for all $j \in \mathbb{N}$ and $K \subset \mathbb{R}^n$ compact. But Theorem 1.1.11 says that the number of seminorms should be countable. The set of seminorms can be made countable by picking only compacts K of the form $[-j, j]^n$. The resulting topology is the same.

Definition 1.2.6 (Support)**.** Let φ be a function $\mathbb{R}^n \to \mathbb{C}$. The support $\operatorname{supp}\varphi$ of φ, is the smallest closed set such that $\varphi = 0$ outside of $\operatorname{supp}\varphi$. That is

$$\operatorname{supp}\varphi = \bigcap_{\substack{\Omega\subseteq\mathbb{R}^n \text{ open}, \\ \varphi=0 \text{ on } \Omega}} \Omega^c.$$

Definition 1.2.7 (Smooth functions of compact support)**.** The space of smooth functions of compact support, denoted $C_c^\infty(\mathbb{R}^n)$ or $\mathcal{D}(\mathbb{R}^n)$, is

$$\{\varphi \in C^\infty(\mathbb{R}^n) : \operatorname{supp}\varphi \text{ is compact}\}.$$

We topologise $\mathcal{D}(\mathbb{R}^n)$ with the LF-topology of smooth convergence on compacts. This means the net $\{\varphi_\lambda\}_{\lambda\in I}, \varphi_\lambda \in \mathcal{D}(\mathbb{R}^n)$ converges in $\mathcal{D}(\mathbb{R}^n)$ to φ when there is a compact $K \subset \mathbb{R}^n$ with $\operatorname{supp}\varphi_\lambda \subseteq K$ (for $\lambda \geq \lambda_0$) and $|\varphi_\lambda - \varphi|_{j,K} \to 0$ for all $j \in \mathbb{N}$ as $\lambda \to \infty$.

Definition 1.2.8 (Schwartz space)**.** We define the family of seminorms by

$$|\varphi|_k := \max_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} (1 + |x|)^k |\partial^\alpha \varphi(x)|, \tag{1.6}$$

for any $k \in \mathbb{N}$. The space of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ is the Fréchet space of smooth functions induced by seminorms (1.6).

It is clear that every test function is a Schwartz functions ($\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$). Additionally, the topologies of $\mathcal{S}(\mathbb{R}^n)$ are such that the inclusion mapping is continuous. This means that if the net $\{\varphi_\lambda\}_{\lambda\in I}$ converges to φ in $\mathcal{D}(\mathbb{R}^n)$, then $||\varphi_\lambda - \varphi||_k \to 0$ for all seminorms $|\cdot|_k$ defined by (1.6). Every Schwartz function is smooth, $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$, and the inclusion mapping is continuous. The property

$$\sup_{x\in\mathbb{R}^\mathbb{K}} (1 + |x|)^k \varphi(x) < \infty \quad \text{for all } k \in \mathbb{N},$$

is called 'rapidly decreasing'. All derivatives of a Schwartz function are thus rapidly decreasing. Now we are ready to define distributions.

1.3 Distributions

Distributions are spaces or continuous linear mappings on function spaces.

Definition 1.3.1 (Linear mapping)**.** Let $\mathbb{K}$ be a field. Let X and Y be vector spaces over $\mathbb{K}$. A map $\phi : X \to Y$ is $\mathbb{K}$-linear if

$$\forall x,\, y \in X,\, \forall \lambda,\, \mu \in \mathbb{K} :\, f(\lambda x + \mu y) = \lambda f(x) + \mu f(y).$$

For us we will always condider either the case $\mathbb{K} = \mathbb{R}$ or the case $\mathbb{K} = \mathbb{C}$. So further let $\mathbb{K}$ represent $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.3.2 (Continuous dual space)**.** Let X be a topological vectorspace. The continuous dual X' of X is

$$\{f : X \to \mathbb{K} :\, f \text{ is a continuous linear mapping}\}.$$

We write the image of a test function $\varphi \in X$ under the functional f as $\langle f, \varphi\rangle$, or when we need to specify the test- and functional space as ${}_{X'}\langle f, \varphi\rangle_X$.

We will refer to elements of $\mathcal{D}'(\mathbb{R}^n)$ as distributions. Elements of $\mathcal{S}'(\mathbb{R}^n)$ are called tempered distributions. Elements of $\mathcal{E}'(\mathbb{R}^n)$ are the distributions of compact support, see Definition 1.3.10 and Theorem 1.3.11. As spaces of linear mappings, these dual spaces are closed under addition and scalar multiplication. We have the reverse of of test space inclusions $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$. Let's describe the topolgy of these spaces.

Definition 1.3.3. Let X be a TVS. A set $B \subseteq X$ is bounded if

$$\forall V \in \mathcal{N}_0,\, \exists r > 0 :\, B \subseteq rV.$$

Definition 1.3.4 (Strong dual topology)**.** Let X be a TVS and put

$$\mathcal{B} = \{B \subseteq X,\, B \text{ is bounded}\}\,.$$

We then define the seminorms for $u \in X'$

$$|u|_B = \sup_{x\in B} \left|{}_{X'}\langle u, x\rangle_X\right|,$$

for any bounded set B of X. The strong topology is induced by the convergence of nets in the seminorms $|\cdot|_B$ as B ranges in $\mathcal{B}$.

We mention another common topology on dual spaces

Definition 1.3.5 (Weak dual topology)**.** Let X be a TVS and let X' be its dual. A net $\{u_\lambda\}_{\lambda\in I} \subset X'$ converges weakly to $u \in X'$ if

$$\langle u_\lambda, \varphi\rangle \to \langle f, \varphi\rangle, \quad \text{as } \lambda \to \infty,$$

for all $\varphi \in X$.

The next theorem connects the weak and strong topology.

Theorem 1.3.6. *Let X be $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ or $\mathcal{E}(\mathbb{R}^n)$, then a sequence $\{u_n\}_{n\in\mathbb{N}} \subset X'$ converges strongly in X' if and only if it converges weakly in X'.*

We consider the distibution spaces $\mathcal{E}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ with the strong topology. On normed spaces, the strong topology reduces to the usual dual norm.

Definition 1.3.7. Let X be a normed space. It's dual X' has the norm

$$||x'||_{X'} := \sup_{\substack{x\in X,\\ ||x||_X=1}} \left|\langle x', x\rangle\right|, \quad x' \in X'.$$

There are some useful inclusions of functions into these distributional spaces.

Example 1.3.8. 1) Let $f \in L^1_{loc}(\mathbb{R}^n)$ be locally absolutely integrable, then the linear functional defined by

$$\langle f, \varphi\rangle = \int_{\mathbb{R}^n} f\varphi dx,$$

is a distribution. A distribution which can be represented by a $L^1_{\text{loc}}(\mathbb{R}^n)$ function is called a regular distribution.

2) Let μ be a measure of locally bounded variation, then similarly

$$\langle \mu, \varphi\rangle = \int_{\mathbb{R}^n} \varphi d\mu,$$

defines a distribution.

3) Let $f \in L^p(\mathbb{R}^n)$, then $f \in \mathcal{D}'(\mathbb{R}^n)$ is a regular distribution and additionally $f \in \mathcal{S}'(\mathbb{R}^n)$.

△

A regular distribution can be uniquely represented by a distribution in the following sense.

Theorem 1.3.9 (Fundamental lemma of calculus of variations)**.** *Let $f, g \in L^1_{\text{loc}}(\mathbb{R}^n)$. If*

$$\int_{\mathbb{R}^n} f\varphi = \int_{\mathbb{R}^n} g\varphi, \quad \forall\varphi \in \mathcal{D}(\mathbb{R}^n),$$

then $f = g$ almost everywhere.

We define the suppport of a distribution by duality as follows.

Definition 1.3.10 (Support of distributions)**.** The support of $u \in \mathcal{D}'(\mathbb{R}^n)$ is

$$\operatorname{supp} u = \bigcap_{\substack{\Omega\subseteq\mathbb{R}^n \text{ open},\\ \langle u,\varphi\rangle=0,\, \forall\varphi\in\mathcal{D}(\Omega).}} \Omega^c.$$

Theorem 1.3.11. *Let $u \in \mathcal{D}'(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n)$, then the support* $\operatorname{supp} u$ *of u is compact.*

A useful representation for distributions of compact support is as distributional derivatives of continuous functions, [16, Theorem 9.14, p297].

Theorem 1.3.12. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $u \in \mathcal{E}'(\Omega)$. Then there exists a neighborhood Ω' of Ω with $f \in C_0(\Omega')$ such that*

$$u = \sum_{\alpha \leq r} c_\alpha \partial^\alpha f, \text{ in } \mathcal{E}'(\Omega),$$

for $r \in \mathbb{N}$ and $c_\alpha \in \mathbb{C}$.

Example 1.3.13. We give several basic examples of distributions.

a) The Dirac delta distribution is the map which evaluates the test function at $x = 0$

$$\langle \delta, \varphi \rangle = \varphi(0).$$

We have $\delta \in \mathcal{E}'(\mathbb{R}^n)$.

b) The Heaviside distribution is

$$\langle H, \varphi \rangle = \int_0^\infty \varphi(x) dx.$$

The Heaviside distribution is a regular distribution represented by the Heaviside function

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

c) Similar to the Heaviside is the sign function

$$\operatorname{sign}(x) = H(x) + H(-x).$$

d) The function $f(x) = \dfrac{1}{x}$ is not $L^1_{\text{loc}}(\mathbb{R}^n)$. Nonetheless we can associate a distribution by exploiting the symmetry of the pole at $x = 0$

$$\langle \text{v.p.} \frac{1}{x}, \varphi \rangle = \int_0^\infty \frac{\varphi(x) - \varphi(-x)}{x} dx.$$

e) Define the distributions $\dfrac{1}{x + i0}, \dfrac{1}{x - i0} \in \mathcal{D}(\mathbb{R})$ by

$$\langle \frac{1}{x \pm i0}, \varphi \rangle = \lim_{\varepsilon \to 0^\pm} \langle \frac{1}{x + i\varepsilon}, \varphi \rangle.$$

The limit exists and is given by

$$\frac{1}{x \pm i0} = \text{v.p.} \frac{1}{x} \mp i\pi\delta(x),$$

which follows by the residue theorem as the limit complex contour integrals.

△

Operations on distributions

A lot of common linear operations on functions can be extended to distributions. Some of these operations like the Fourier transform gain more strength in framework of distribution theory.

Definition 1.3.14. Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$. The tensor product $w = u \otimes v \in \mathcal{D}'(\mathbb{R}^{2n})$ is defined as

$$\begin{aligned} \langle w(x,y), \varphi(x,y) \rangle &= \langle u(y), \langle v(x), \varphi(x,y) \rangle \rangle, \\ &= \langle v(x), \langle u(y), \varphi(x,y) \rangle \rangle, \end{aligned}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^{2n})$.

Example 1.3.15. Dirac delta tensor Heaviside:

$$\begin{aligned}\langle \delta \otimes H, \varphi \rangle &= \langle H(y), \langle \delta(x), \varphi(x,y) \rangle \rangle, \\ &= \langle H(y), \varphi(0,y) \rangle, \\ &= \int_0^\infty \varphi(0,y) dy.\end{aligned}$$

△

A large class of operations on distributions are defined by transposition. The simplest example is the derivative.

Definition 1.3.16 (Distributional derivative)**.** The distributional partial derivative $\partial_{x_i} u$ for $u \in \mathcal{D}'(\mathbb{R}^n)$ is defined by

$$\langle \partial_{x_i} u, \varphi \rangle := -\langle u, \partial_{x_i} \varphi \rangle,$$

which coincides with the integration by parts formula for $u \in \mathcal{D}(\mathbb{R}^n)$.

In general for a linear continuous operator $A : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ we seek a transpose operator $A^* : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$, linear and continuous, such that $\langle A\psi, \varphi \rangle = \langle \psi, A^* \varphi \rangle$, $\forall \varphi, \psi \in \mathcal{D}$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{D}'(\mathbb{R}^n)$, the operator A extends to a unique linear, continuous operator $\tilde{A} : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ by the formula

$$\langle \tilde{A} u, \psi \rangle := \langle u, A^* \psi \rangle, \quad \forall \psi \in \mathcal{D}.$$

The approaches for $\mathcal{E}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ are analogous.

Example 1.3.17. a) The Dirac delta is the derivative of the Heaviside.

$$\partial_x H(x) = \delta(x).$$

b) The principle value v.p.$\frac{1}{x}$ is the derivative of the regular distibution $\log |x|$

$$\partial_x \log |x| = \text{v.p.}\frac{1}{x}.$$

c) Let $\alpha \in \mathbb{N}^n$ be a multi-index. We define $\delta^{(\alpha)}(x)$ as the α-th distributional derivative of δ,

$${}_{\mathcal{E}'(\mathbb{R})}\langle \delta^{(\alpha)}, \varphi \rangle_{\mathcal{E}(\mathbb{R})} := (-1)^{|\alpha|} \partial_x^\alpha \varphi(0),$$

△

In [34], Schwartz defined the multiplication of smooth function and a distribution.

Definition 1.3.18 (Schwartz product)**.** Let $f \in C^\infty(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$ then we define their Schwartz product fu by

$$\langle fu, \varphi \rangle = \langle u, f\varphi \rangle.$$

This well-known product is the starting point in our investigation of a product of distributions. For tempered distributions there is an analogous product. Instead of $C^\infty(\mathbb{R}^n)$ we can multiply by $\mathcal{O}_M(\mathbb{R}^n)$.

Definition 1.3.19. Write $\mathbb{R}[x]$ for the set of polynomials with real coefficients. The functions of slow growth $\mathcal{O}_M(\mathbb{R}^n)$ consists of the smooth functions f that satisfy

$$\forall \alpha \in \mathbb{N}^n, \, \exists p(x) \in \mathbb{R}[x], \, \exists A \in \mathbb{R} : \sup_{\substack{x \in \mathbb{R}^n, \\ |x| \geq A}} |\partial_x^\alpha f(x)| \leq p(x).$$

That means all derivatives are polynomially bounded for large $|x|$.

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{O}_M(\mathbb{R}^n)$, the derivatives of any order of $f\varphi$ are again rapidly decreasing. It remains to define

$$\langle fu, \varphi \rangle = \langle u, f\varphi \rangle,$$

for $u \in \mathcal{S}'$ and $f \in \mathcal{O}_M$.

We start first by defining the Fourier transform of an L^1 function.

Definition 1.3.20 (Fourier transform)**.** For $f \in L^1(\mathbb{R}^n)$ we define its Fourier transform $\mathcal{F}(f)$ as

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx,$$

where $x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n$ is the dot product of the vectors x and ξ. Alternatively we use the notation $\hat{f} = \mathcal{F}f$.

We will also use the notation $\mathcal{F}(f) = \hat{f}$. It is important to note that there exist multiple conventions for the definition of the Fourier transform. Definition (1.3.20) is the unitary Fourier transform with ordinary frequency.

Theorem 1.3.21 (Riemann-Lebesgue lemma)**.** *If $f \in L^1(\mathbb{R}^n)$, then $\mathcal{F}(f)$ is continuous and*

$$\lim_{|\xi|\to\infty} \mathcal{F}(f)(\xi) = 0.$$

Definition 1.3.22. The inverse Fourier transform $\mathcal{F}^{-1}$ is

$$\mathcal{F}^{-1}(f)(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} f(x)dx, \quad \text{for } f \in L^1.$$

The inverse Fourier transform satisfies $\mathcal{F}^{-1}(f)(\xi) = \mathcal{F}(f)(-\xi)$.

Theorem 1.3.23 (Fourier inversion theorem)**.** *If $f \in L^1(\mathbb{R}^n)$ and $\mathcal{F}(f) \in L^1(\mathbb{R}^n)$, then $\mathcal{F}^{-1}(\mathcal{F}(f))(x) = f(x)$.*

We work towards extending the Fourier transform to tempered distributions.

Theorem 1.3.24. *If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R}^n)$ and the map $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is continuous.*

Theorem 1.3.25. *The Fourier transform has an extension to the the space $L^2(\mathbb{R}^n)$ (also denoted $\mathcal{F}$). If $f \in L^2(\mathbb{R}^n)$, then $\mathcal{F}(f) \in L^2(\mathbb{R}^n)$ and the map $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous.*

Theorem 1.3.26 (Parseval's theorem)**.** *Let f, $g \in L^2(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \mathcal{F}(f)(x)g(x)dx = \int_{\mathbb{R}^n} f(x)\mathcal{F}(g)(x)dx.$$

Notably, Parseval's theorem holds for Schwartz functions. Thus the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ is its own transpose. We use this to extend the Fourier transform to tempered distributions.

Definition 1.3.27. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then the Fourier transform $\mathcal{F}(f)$ is defined by

$$\langle \mathcal{F}(f), \varphi\rangle = \langle f, \mathcal{F}(\varphi)\rangle, \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Due to continuity of the Fourier transform $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, composition with the continuous linear functional $f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ gives again continuous linear functional $\mathcal{F}(f) : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : g \mapsto f(\mathcal{F}(g))$.

Example 1.3.28. We calculate some Fourier transforms.

a) The Fourier transform of the Dirac delta is 1, i.e.

$$\mathcal{F}\delta = 1.$$

To see this, we just need to apply the definition. We can then calculate

$$\begin{aligned}
\langle \mathcal{F}\delta, \varphi\rangle &= \langle \delta, \mathcal{F}\varphi\rangle, \\
&= \mathcal{F}\varphi(0) \\
&= \int_{\mathbb{R}^n} e^{-2\pi i x\cdot 0}\varphi(x)dx, \\
&= \int_{\mathbb{R}^n} \varphi(x)dx, \\
&= \langle 1, \varphi\rangle.
\end{aligned}$$

b) The Fourier transform of the Heaviside is given by

$$\mathcal{F}H(x) = \frac{1}{2\pi i(x - i0)}.$$

To see this we write the Heaviside as the limit of exponentials on $[0, +\infty)$

$$\begin{aligned} {}_{\mathcal{S}'(\mathbb{R})}\langle \widehat{H}, \varphi \rangle_{\mathcal{S}(\mathbb{R})} &= {}_{\mathcal{S}'(\mathbb{R})}\langle H, \widehat{\varphi} \rangle_{\mathcal{S}(\mathbb{R})} \\ &= \int_0^{+\infty} \widehat{\varphi}(x)dx \\ &= \lim_{\varepsilon \to 0^+} \int_0^{+\infty} e^{-2\pi\varepsilon} \widehat{\varphi}(x)dx. \end{aligned}$$

This limit is valid because the integral of $\widehat{\varphi}$ on $(R, +\infty)$ goes to zero as $R \to 0$ and $e^{-2\pi\varepsilon}$ converges uniformly to 1 on the compacts $[0, R]$. Thus we have

$$\begin{aligned} \langle \widehat{H}, \varphi \rangle &= \lim_{\varepsilon \to 0^+} \int_0^{+\infty} e^{-2\pi\varepsilon} \int_{-\infty}^{+\infty} e^{-2\pi i x \xi} \varphi(\xi) d\xi dx \\ &= \lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} \varphi(\xi) \left. \frac{e^{-2\pi i x \xi - 2\pi\varepsilon}}{-2\pi i \xi - 2\pi\varepsilon} \right|_{x=0}^{x=+\infty} d\xi \\ &= \lim_{\varepsilon \to 0^+} \int_{-\infty}^{+\infty} \frac{\varphi(\xi)}{2\pi i(\xi - i\varepsilon)} d\xi \\ &= \frac{1}{2\pi i} \langle \frac{1}{x - i0}, \varphi(x) \rangle. \end{aligned}$$

△

Definition 1.3.29 (Convolution). Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$. The convolution $w = u * v \in \mathcal{D}'(\mathbb{R}^n)$ is defined as

$$\langle w, \varphi \rangle := {}_{\mathcal{D}'(\mathbb{R}^{2n})}\langle u(x)v(y), \varphi(x + y) \rangle_{\mathcal{E}(\mathbb{R}^{2n})}, \quad \text{for } \varphi \in \mathcal{D}(\mathbb{R}^n),$$

if the righthandside is well-defined.

Since the convolution is only partially defined $\mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$, it is useful to consider the restriction to a pair of subspaces $A \times B$, where the convolution is now defined on the whole domain. For a specific choice of subspaces we can then exploit stronger properties.

Proposition 1.3.30. *a) Let $u \in \mathcal{E}'(\mathbb{R}^n)$ and $v \in \mathcal{D}'(\mathbb{R}^n)$, then the convolution $u * v \in \mathcal{D}'(\mathbb{R}^n)$ exists.*

b) If $u \in \mathcal{S}(\mathbb{R}^n)$ and $v \in \mathcal{O}_M(\mathbb{R}^n)$, then $u * v \in \mathcal{O}_M(\mathbb{R}^n)$.

c) If $u \in \mathcal{S}(\mathbb{R}^n)$ and $v \in \mathcal{E}'(\mathbb{R}^n)$, then $u * v \in \mathcal{S}(\mathbb{R}^n)$.

d) If $u \in \mathcal{E}(\mathbb{R}^n)$ and $v \in \mathcal{E}'(\mathbb{R}^n)$, then $u * v \in \mathcal{E}(\mathbb{R}^n)$.

Definition 1.3.31. The space of convolutors $O'_C(\mathbb{R}^n)$ consists of the Schwartz distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\mathcal{F}(u) \in \mathcal{O}_M(\mathbb{R}^n).$$

Proposition 1.3.32. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$, $v \in \mathcal{O}'_C(\mathbb{R}^n)$, then $u * v$ is in $\mathcal{S}'(\mathbb{R}^n)$.*

This includes the special case $v \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{O}'_C(\mathbb{R}^n)$. For rapidly decreasing functions, the Fourier transform and the convolution interact.

Theorem 1.3.33 (Exchange formula). *For all $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g). \tag{1.7}$$

A generalised exchange formula is the basis of the Fourier product in section 2.5. The Dirac delta is the unit element for the convolution.

Example 1.3.34. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ then the convolution $u * \delta$ always exists and equals

$$u * \delta = u.$$

The proof follows by a direct computation,

$$\begin{aligned}\langle u * \delta, \psi \rangle &= \langle u(x) \langle \delta(y), \varphi(x+y) \rangle \rangle, \\ &= \langle u(x), \varphi(x) \rangle.\end{aligned}$$

△

Regularisation and delta nets

Definition 1.3.35. Let $u \in \mathcal{D}'(\mathbb{R}^n)$. The net $\{f_\lambda\}_{\lambda \in I} \subset \mathcal{D}'(\mathbb{R}^n)$ of functions $f_\lambda : \mathbb{R}^n \to \mathbb{C}$ is a regularisation of u if

$$\lim_{\lambda \in I} f_\lambda \to u, \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

Regularisation is often used for nets of more regular functions, e.g. $f_\lambda \in C^\infty(\mathbb{R}^n)$. With an abuse of language, we will also call any element f_λ of a fixed regularisation $\{f_\lambda\}_{\lambda \in I}$ of $u \in \mathcal{D}'(\mathbb{R}^n)$, a regularisation of u. The element f_λ is then to be interpreted as an approximation of u.

A regularisation of the Dirac delta distribution is called a delta net. There are different classes of delta nets which deserve special attention. One of the more broad classes is the class of strict delta nets.

Definition 1.3.36 (Strict delta net)**.** A strict delta net is a net of test functions $(\rho^\varepsilon)_{\varepsilon \in (0,1]} \subset \mathcal{D}(\mathbb{R}^n)$ with the properties

$$\begin{aligned}\operatorname{supp}(\rho^\varepsilon) \to \{0\}, &\quad \text{when } \varepsilon \to 0, \\ \int_{\mathbb{R}^n} \rho^\varepsilon(x)dx = 1, &\quad \text{for all } \varepsilon \in (0,1], \\ \int_{\mathbb{R}^n} |\rho^\varepsilon(x)|dx \text{ is uniformly bounded for } &\varepsilon \in (0,1].\end{aligned}$$

Every strict delta net is indeed a delta net.

Theorem 1.3.37. *Every strict delta net $(\rho^\varepsilon)_{\varepsilon \in (0,1]}$ converges to $\delta(x)$ in $D'(\mathbb{R}^n)$ as $\varepsilon \to 0$.*

Proof. For all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\begin{aligned}\left|\langle \rho^\varepsilon(x) - \delta(x), \varphi \rangle\right| &= \left| \int_{\mathbb{R}^n} \rho^\varepsilon(x)(\varphi(x) - \varphi(0))dx \right|, && (1.8) \\ &\leq \sup_{x \in \operatorname{supp} \rho^\varepsilon} \left|\varphi(x) - \varphi(0)\right| \int_{\mathbb{R}^n} \left|\rho^\varepsilon(x)\right| dx, && (1.9) \\ &\leq \operatorname{diam}(\operatorname{supp}(\rho^\varepsilon)) \sum_{i=1}^n \sup_{x \in \mathbb{R}^n} \left|\partial_{x_i}\varphi(x)\right| \int_{\mathbb{R}^n} \left|\rho^\varepsilon(x)\right| dx, && (1.10) \\ &\leq \operatorname{diam}(\operatorname{supp} \rho^\varepsilon) C |\varphi|_{1,K}, && (1.11)\end{aligned}$$

with the constant $C = n \int_{\mathbb{R}^n} \left|\rho^\varepsilon(x)\right| dx$ and the diameter of a metrizable set U:

$$\operatorname{diam}(U) = \sup_{x,y \in U} d(x-y), \quad \text{with } d \text{ the metric of } U.$$

The estimate (1.11) is uniform on bounded sets of $\mathcal{D}(\mathbb{R}^n)$. Therefore $\rho^\varepsilon \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$. □

As we will see along this thesis regularisation and passage to the limit is very useful to extend non-linear operations from functions to distributions. It is one of the main ways to define a product of distributions. The very weak solution concept also relies on regularisation.

1.4 Sobolev spaces

Sobolev spaces and the weak formulation are used in the analysis of partial differential equations. Main references are [1, 8, 11].

Sobolev spaces combine the concepts of L^p-spaces and of distributional derivatives. In this context we ask that the distributional derivative is again a regular distribution. We call it the weak derivative.

Definition 1.4.1 (Weak derivative)**.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We call $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ the α-th weak derivative of f if

$$\int_{\mathbb{R}^n} g\varphi = (-1)^\alpha \int_{\mathbb{R}^n} f \partial^\alpha \varphi,$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

An equivalent formulation is: $g \in \mathcal{D}'(\mathbb{R}^n) \cap L^1_{\text{loc}}(\mathbb{R}^n)$ is the α-the distributional derivative of the regular distribution $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Definition 1.4.2 (Sobolev space)**.** For $1 \leq p < \infty$ and $m \in \mathbb{N}$, for $f : \mathbb{R}^n \to \mathbb{C}$ define the norm

$$||f||_{W^{m,p}(\mathbb{R}^n)} := \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} ||\partial^\alpha f||_{L^p(\mathbb{R}^n)}.$$

For $p = \infty$, $m \in \mathbb{N}$ we define

$$||f||_{W^{m,\infty}(\mathbb{R}^n)} := \max_{\alpha \in \mathbb{N}^n, |\alpha| \leq m} ||\partial^\alpha f||_{L^\infty(\mathbb{R}^n)}.$$

The Sobolev space $W^{m,p}(\mathbb{R}^n)$ is the normed space of functions $\mathbb{R}^n \to \mathbb{C}$ with finite $W^{m,p}(\mathbb{R}^n)$-norm.

Sobolev spaces thus consist of L^p functions such that all derivatives up to order m are also in L^p. When $p = 2$ we use the notation $W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n)$.

Definition 1.4.3. The space $W^{m,p}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. That is, a function $f \in W^{m,p}(\Omega)$ is in $W^{m,p}_0(\Omega)$ if there is a sequence $\{\varphi_i\}_{i \in \mathbb{N}} \subset C_0^\infty(\Omega)$ such that

$$\lim_{i \to \infty} ||f - \varphi_i||_{W^{m,p}(\Omega)} = 0.$$

These spaces are interpreted as the functions $f \in W^{m,p}(\Omega)$ such that

$$\partial^\alpha f = 0, \text{ on the boundary } \partial\Omega \quad \text{for } |\alpha| \leq m - 1,$$

Again we use the special notation $W^{m,2}_0(\Omega) = H^m_0(\Omega)$.

Definition 1.4.4. For $1 \leq p \leq \infty$ and $\ell \in \mathbb{R}_-$ we define negative order Sobolev spaces using weak derivatives

$$W^{\ell,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{D}'(\mathbb{R}^n) : u = \sum_{|\alpha| \leq |\ell|} \partial^\alpha u_\alpha, \ u_\alpha \in L^p(\mathbb{R}^n) \right\}.$$

Negative order Sobolev spaces are the dual spaces of the positive order Sobolev spaces. That is $W^{-m,q}(\Omega)$ is dual to $W^{m,p}(\Omega)$ if $1 \leq p < \infty$, $m \in \mathbb{N}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $v \in W^{m,p}(\Omega)$ and $u \in W^{-m,q}(\Omega)$ ($m \in \mathbb{N}, 1 \leq p, q \leq \infty, \frac{1}{p} + \frac{1}{q} = 1$). The duality is explicitly described by applying distributional derivatives formally, then using $L^p - L^q$ duality. Writing u as

$$u = \sum_{|\alpha| \leq |m|} \partial^\alpha u_\alpha, \quad \text{with } u_\alpha \in L^q(\Omega).$$

gives

$${}_{W^{-m,q}(\Omega)}\langle u, v \rangle_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq |m|} {}_{W^{-m,q}(\Omega)}\langle \partial^\alpha u_\alpha, v \rangle_{W^{m,q}(\Omega)} = \sum_{|\alpha| \leq |m|} {}_{L^q(\Omega)}\langle u_\alpha, \partial^\alpha v \rangle_{L^p(\Omega)}.$$

As Sobolev functions are differentiable (in the weak sense), one would expect for the Leibniz rule to hold. However only in certain cases we can guarantee this. The next theorem is from [8, Theorem 9.4].

Theorem 1.4.5 (Derivative of a Sobolev product)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $1 \leq p \leq \infty$. Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and,*

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i}, \quad i = 1, 2, \ldots, n.$$

Proof. There exist sequences $\{u_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^n)$ such that,

$$u_{n|\Omega} \to u, \qquad v_{n|\Omega} \to v \qquad \text{in } L^p(\Omega) \text{ and a.e. on } \Omega, \tag{1.12}$$
$$\nabla u_{n|\omega} \to \nabla u_{|\omega}, \qquad \nabla v_{n|\omega} \to \nabla v_{|\omega} \qquad \text{in } L^p(\omega)^n \text{ for all } \omega \subset \Omega \text{ compact.} \tag{1.13}$$

Additionally we can ask that

$$\|u_n\|_{L^\infty(\mathbb{R}^n)} \leq \|u\|_{L^\infty(\Omega)} \quad \text{and} \quad \|v_n\|_{L^\infty(\mathbb{R}^n)} \leq \|v\|_{L^\infty(\Omega)}. \tag{1.14}$$

The construction of the sequences $\{u_n\}_{n\in\mathbb{N}}$ is done via convolution with a sequence of mollifiers ρ_n. Then we set $u_n = \rho_n * u$. The convergence (1.12) and the bounds (1.14) follow from Young's convolution inequality $||\rho_n * u||_{L^p} \leq ||\rho_n||_{L^\infty} ||u||_{L^p}$ On the other hand,

$$\int_\Omega u_n v_n \partial \partial x_i \varphi = -\int_\Omega \left(\partial_{x_i} u_n v_n + u_n \partial_{x_i} v_n\right) \varphi \quad \forall \varphi \in C_c^1(\Omega).$$

Passing to the limit, by dominated convergence, this becomes,

$$\int_\Omega uv \partial_{x_i} v = -\int_\Omega \left(\partial_{x_i}(u) v + u \partial_{x_i} v\right) \varphi \quad \forall \varphi \in C_c^1(\Omega).$$

□

For the analysis of time-dependent PDEs we need definitions for vector valued functions.

Vector valued functions

For weak solutions of time-dependent problems vector valued functions are often used. We refer to Dautray [11, chapter 18 p496]. Consider a Banach space X with norm $||\cdot||_X$. Vector valued function spaces are again normed.

Definition 1.4.6. For $1 \leq p < \infty$ we define the space $L^p((0,T), X)$ of functions $u : [0,T] \to X$ such that

$$||u||_{L^p((0,T),X)} = \left(\int_0^T ||u(t)||_X^p dt\right)^{\frac{1}{p}} < \infty.$$

The space $C([0,T], X)$ of continuous functions $u : [0,T] \to X$ such that

$$||u||_{C([0,T],X)} = \max_{t\in[0,T]} ||u(t)||_X < \infty.$$

The space $L^\infty((0,T), X)$ of functions $u : [0,T] \to X$ such that

$$||u||_{L^\infty((0,T),X)} = \inf\{B : ||u(t)||_X \leq B \text{ for almost all } t \in [0,T]\} < \infty.$$

Additionally we have Sobolev type spaces.

Definition 1.4.7. Let $1 \leq p < \infty$ and $m \in \mathbb{N}$ and X a Banach space. The space $W^{m,p}((0,T), X)$ is the normed space of functions $u : [0,T] \to X$ such that

$$||u||_{W^{m,p}((0,T),X)} = \left(\int_0^T \sum_{\alpha\in\mathbb{N},\alpha\leq m} ||\partial_t^\alpha u(t)||_X^p dt\right)^{\frac{1}{p}} < \infty.$$

Again we will use the notation $H^m((0,T), X)$ for the case $m = 2$.

1.5 Weak formulation and finite element method

Like the weak derivative, Definition 1.4.1, the weak formulation interprets the PDE in a variational sense, Theorem 1.3.9. Standard works on partial differential equations with discussion of variational formulations are Evans [15] and Dautray-Lions [11]. We introduce the weak solution concept by an example of a one-dimensional differential equation. Consider the following boundary value problem for $f \in C((0,1))$:

$$\begin{cases} u(x) - u''(x) = f(x), & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.15}$$

There exists a unique $u \in C^2((0,1))$ that solves (1.15). The function u is called a classical solution since the derivatives occuring in equation (1.15) exist in the classical sense and are continuous. Interpreting (1.15) as equality of regular distributions gives

$$\int_0^1 u(x)\varphi(x)dx - \int_0^1 u''(x)\varphi(x)dx = \int_0^1 f(x)\varphi(x)dx, \quad \forall \varphi \in \mathcal{D}((0,1)).$$

Further, integration by parts gives

$$\int_0^1 u(x)\varphi(x)dx - u'(1)\varphi(1) + u'(0)\varphi(0) + \int_0^1 u'(x)\varphi'(x)dx = \int_0^1 f(x)\varphi(x)dx, \quad \forall \varphi \in \mathcal{D}((0,1)). \tag{1.16}$$

We weak formulation extends the classical formultation by allowing u to be in a larger class of functions than $C^2((0,1))$. Equation (1.16) is well-defined for $u, \varphi \in H^1((0,1))$. By the boundary conditions of (1.15) we restrict ourselves further to $V = H_0^1((0,1))$. This has the side effect that the terms $-u'(1)\varphi(1) + u'(0)\varphi(0)$ vanish. The result is the weak formulation of the boundary value problem (1.15). For $f \in L^2((0,1))$, find $u \in H_0^1((0,1))$ such that

$$\int_0^1 u(x)\varphi(x)dx + \int_0^1 u'(x)\varphi'(x)dx = \int_0^1 f(x)\varphi(x)dx, \quad \forall \varphi \in H_0^1((0,1)).$$

It is useful to write the weak formulation in terms of a bilinear form and a linear functional. Define the bilinear form $a : H_0^1((0,1)) \times H_0^1((0,1)) \to \mathbb{C}$ by

$$a(u,v) = \int_0^1 u(x)v(x)dx + \int_0^1 u'(x)v'(x)dx, \quad u, v \in H_0^1((0,1)),$$

and the linear functional $f : H_0^1((0,1)) \to \mathbb{C}$ by

$$f(v) = \int_0^1 f(x)\varphi(x)dx, \quad v \in H_0^1((0,1)).$$

Then we can write the weak formulation as $a(u,\varphi) = f(\varphi)$. An existence theorem then depends solely on the properties of a and f. Most famous is the Lax-Milgram theorem [9, Theorem 1.1.13] which provides conditions under which the solution exists and is unique.

Theorem 1.5.1 (Lax-Milgram). *Let V be a Hilbert space and $a(\cdot,\cdot)$ a bilinear form on V. Suppose that a is bounded*

$$|a(u,v)| \leq C||u||_V||v||_V,$$

and V-elliptic

$$a(u,u) \geq c||u||_V^2.$$

Then for any $f \in V'$, there is a unique solution $u \in V$ to

$$a(u,v) = f(v), \quad \forall v \in V.$$

Additionaly we have the energy estimate

$$||u||_V \leq \frac{1}{c}||f||_{V'}. \tag{1.17}$$

Finite element method

The finite element method (FEM) is a method for solving differential equations numerically. The FEM is unique among numerical methods since it interprets the boundary value problem by the weak formulation. For a time-dependent weak formulation we will later introduce use a semi-discretisation scheme.

Consider the weak formulation of a boundary value problem. Find $u \in V$ such that

$$a(u, v) = f(v), \quad \forall v \in V, \tag{1.18}$$

for some bilinear form a and linear form f. We approximate equation (1.18) to a subspace V_h of V. Let $u_h \in V_h$ satisfy

$$a(u_h, v) = f(v), \quad \forall v \in V_h. \tag{1.19}$$

We call u_h the Galerkin approximation of u for the space V_h.

Theorem 1.5.2 (Céa's lemma)**.** *Let V be a normed space with norm $||\cdot||_V$ and let $V_h \leq V$ a normed subspace. Let $a : V \times V \to \mathbb{C}$ be a continuous and V-elliptic bilinear form and $f : V \to \mathbb{C}$ a continuous linear form. Suppose that $u \in V$ satisfies* (1.18) *and that $u_h \in V_h$ satisfies* (1.19)*, then*

$$||u - u_h||_V \leq C \inf_{v \in V_h} ||u - v||_V.$$

To improve the quality of the Galerkin approximation we thus need to choose better and larger subspaces V_h. In the finite element method the spaces V_h are given by finite element spaces.

Definition 1.5.3. Let $n \in \mathbb{N}$ be the number of midpoints and put $h = \dfrac{1}{n+1}$ the distance between midpoints. Split the interval $[a, b]$ into $n+1$ intervals of equal length $I_j = [jh, (j+1)h]$, $j = 0 \dots n$. The finite element space $V^{m,k}$ subject to the intervals $I_1, \dots I_n$ is defined by

$$V_h^{k,m} = \{f \in C^m([a, b]) : f|_{I_j} \text{is a polynomial of degree } k, \, j = 0, \dots, n\}.$$

The spaces $V_h^{m,k}$ are finite-dimensional. This means problem (1.19) reduces to a matrix equation. Implementations of the finite element method exploits a good choice of basis functions to efficiently solve these matrix equations.

Chapter 2

Multiplication of distributions

The main part of this chapter concerns a theoretical discussion of intrinsic multiplication of distributions as presented in Oberguggenberger's 'Multiplication of distributions and applications to partial differential equations' [29, Chapter I, II]. Near the end of the chapter we discuss extrinsic multplication with Schwartz's impossibility result and Colombeau's algebra. Lastly we present the very weak solution concept for partial differential equations.

2.1 Introduction

Before we discuss what products of distributions are possible, let us investigate through examples how a multplication of distributions should work.

The multiplication problem for functions

Consider the case of the product of two functions $u, v : \mathbb{R} \to \mathbb{C}$. We can easely define a pointwise multiplication $w(x) = u(x)v(x)$. The map w is again a function $\mathbb{R} \to \mathbb{C}$, which is in the class of functions we started with.

Now consider two continuous functions $f, g : \mathbb{R} \to \mathbb{C}$. Using the pointwise multiplication, the product h is continuous. Similarly the product h of $f, g \in C^k(\mathbb{R})$ or $f, g \in C^\infty(\mathbb{R})$ is again $C^k(\mathbb{R})$ or $C^\infty(\mathbb{R})$ respectively.

Let $f, g \in L^\infty(\mathbb{R})$, then the product (pointwise almost everywhere) $h = fg$ is in $L^\infty(\mathbb{R})$ since we have the bound $||h||_{L^\infty(\mathbb{R})} \leq ||f||_{L^\infty(\mathbb{R})}||g||_{L^\infty(\mathbb{R})}$.

The above products are regular and intrisic. This means every product fg, for $f, g \in X$, is defined, in this case as a map $\mathbb{R} \to \mathbb{C}$, and the product is in the original function space X. It is clear that pointwise multiplication of functions $\mathbb{R}^n \to \mathbb{C}$ is always regular, but on some important functions spaces, it is not intrinsic.

Let $X = L^1(\mathbb{R})$. Then the pointwise product is not intrinsic. For example take $f, g \in L^1(\mathbb{R})$

$$f(x) = g(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & \text{if } |x| \leq 1, \\ \dfrac{1}{x^2}, & \text{if } |x| \geq 1, \end{cases}$$

then the product (pointwise almost everywhere) $h = fg$ is

$$h(x) = f(x)g(x) = \begin{cases} \dfrac{1}{x}, & \text{if } |x| \leq 1, \\ \dfrac{1}{x^4}, & \text{if } |x| \geq 1. \end{cases}$$

The product h is not in $L^1(\mathbb{R})$ as the pole at $x = 0$ is not absolutely integrable. The same phenomenon happens for $f, g \in L^p(\mathbb{R}), (1 \leq p < \infty)$. The pointwise multiplication does not respect the L^1-requirement of absolute integrability.

We see that the problem is local. One solution is to ask boundedness. So consider $f, g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then the product is again in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. We lost functions that behave locally like x^r $(r < -1)$, but we kept the global integrability requirement.

Within $L^1(\mathbb{R})$, we can consider the case where $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The pointwise product fg is then in $L^1(\mathbb{R})$. This product cannot be extended to a multiplication map $L^1(\mathbb{R}) \times L^1(\mathbb{R}) \to L^1(\mathbb{R})$. We call it a partial multiplication on $L^1(\mathbb{R})$.

We compare the situation of functions to that of distributions. Every continuous linear map between TVS is bounded. For distributions this means: for each $K \subseteq \mathbb{R}^n$ compact there exist $N_K \in \mathbb{N}, M_K \in \mathbb{R}$ such that

$$|\langle u, \varphi \rangle| \leq M_K |\varphi|_{N_K, K}, \quad \text{for all } \varphi \in \mathcal{D}(K).$$

Distributions thus satisfy a boundedness condition. The question is now whether a product of distributions will violate this boundedness condition. We review some examples where the product of distributions fails.

Example 2.1.1 (The square of the Dirac delta)**.** We consider a delta net $(\varphi_\varepsilon)_{\varepsilon \in (0,1]}$, definition 1.3.36, as a regularisation of the Dirac delta δ. Assume also that φ_ε is real-valued. Now consider the square $\varphi_\varepsilon^2(x)$ of such a regularisation. We hope that this net of squares converges to some element $u \in \mathcal{D}'(\mathbb{R})$, which we can then call δ^2. Choose some $\psi \in \mathcal{D}(\mathbb{R})$ with $\psi(x) = 1$ on $[0, 1]$. Then we have the valuations

$$\int_{\mathbb{R}} \varphi_\varepsilon^2(x)\psi(x)dx = \int_{\mathbb{R}} \varphi_\varepsilon^2(x)dx,$$

for ε small enough. If $\varphi_\varepsilon^2(x)$ converges to the distribution u, then

$$\int_{\mathbb{R}} \varphi_\varepsilon^2(x)dx \to \langle u, \psi \rangle.$$

More specifically φ_ε is bounded in $L^2(\mathbb{R})$ and thus it has a convergent subsequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R})$. But φ_ε converges to δ in $\mathcal{D}'(\mathbb{R})$. This implies that $\varphi_n \to \delta$ in $L^2(\mathbb{R})$, but δ is not in $L^2(\mathbb{R})$. We conclude that the limit of φ_ε^2 does not exist in $\mathcal{D}'(\mathbb{R})$.

Multiplication trough some type of regularisation is one of the most general definitions of multiplication of distributions we have. The fact that taking any one strict delta net leads us to a contradication, speaks of the true impossibility of δ^2 being a distribution. △

One of the most powerful methods for defining products is regularisation. However when applied without caution, it can give rise to a wide range of contradictory results. We show how passing trough the limit for arbitrary regularisations gives the product

$$H(x)\delta(x) = c\delta(x),$$

for different values of $c \in \mathbb{R}$.

Example 2.1.2 (Heaviside times Dirac delta)**.** We consider different ad hoc regularisations of the product. Let $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ be a strict delta net. Let $(H_\varepsilon)_{\varepsilon\in(0,1]}$ be a regularisation of the Heaviside function H.

Assume that $(H_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies $\operatorname{supp} H_\varepsilon \subseteq (\varepsilon, +\infty)$ and that $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies $\operatorname{supp} \varphi_\varepsilon \subseteq [-\varepsilon, \varepsilon]$. Then the product $\varphi_\varepsilon H_\varepsilon$ is indentically zero and passing to the limit gives $\delta H = 0$.

Suppose differently that H_ε is identically one on $[-\varepsilon, +\infty)$, then

$$\varphi_\varepsilon H_\varepsilon = \varphi_\varepsilon \to \delta.$$

Finally let's take the regularisations $\delta_\varepsilon = \varphi_\varepsilon$ and $H_\varepsilon = H * \varphi_\varepsilon$. Then we have

$$\varphi_\varepsilon H_\varepsilon = \varphi_\varepsilon(H * \varphi_\varepsilon) = \frac{d}{dx}\frac{1}{2}(H * \varphi_\varepsilon)^2 \to \frac{d}{dx}\frac{1}{2}H = \frac{1}{2}\delta.$$

This last result will be the one that is produced by the methods later in this chapter. However it is not obvious that $\frac{1}{2}$ is the right constant. For example when we solve the following differential equation

$$\begin{cases} \frac{d}{dt}y(t) = \delta(t)y(t), \\ y(-\infty) = 1. \end{cases}$$

Using the ansatz $y(t) = 1 + \alpha H(t),\ \alpha \in \mathbb{R}$, we get

$$\alpha\delta(t) = \delta(t)(1 + \alpha H(t)) = \delta(t) + \alpha\delta(t)H(t).$$

Using the product $\delta H = \frac{1}{2}\delta$ this is

$$\alpha\delta(t) = (1 + \frac{\alpha}{2})\delta(t),$$

and thus $\alpha = 2$. Now we solve it in a different way. Let $y_\varepsilon(t)$ be the solution to the regularised differential equation

$$\frac{d}{dt}y_\varepsilon(t) = \varphi_\varepsilon(t)y_\varepsilon(t),$$
$$y_\varepsilon(-\infty) = 1.$$

This equation is solved explicitly by $y_\varepsilon(t) = \exp\left(\int_{-\infty}^{t} \varphi_\varepsilon(s)ds\right)$. Passing to the limit, we get

$$y(t) = \lim_{\varepsilon \to 0} \exp\left(\int_{-\infty}^{t} \varphi_\varepsilon(s)ds\right) = \exp(H(t)) = 1 + (e-1)H(t).$$

Assuming a product of the form $\delta(t)H(t) = c\delta(t)$, this would imply

$$c = 1 - \frac{1}{e-1} = \frac{e-2}{e-1}.$$

△

The previous example makes clear that we cannot arbitrarily regularise (partial) differential equations that contain products of distributions. We are in need of a rigorous theory of multiplication of distributions which takes into account stability under (certain classes of) regularisations.

In general the Leibniz rule for the derivative of the product of distributions does not hold. We give an example.

Example 2.1.3 (Breaking the Leibniz rule)**.** Assume the product $\delta(x)H(x) = \frac{1}{2}\delta(x)$. Taking the derivative gives $\partial_x(\delta(x)H(x)) = \frac{1}{2}\delta'(x)$. If the Leibniz rule holds, this should be equal to

$$\partial_x\delta(x)H(x) + \delta(x)\partial_x H(x) = \delta'(x)H(x) + (\delta(x))^2.$$

We showed in Example 2.1.1 that the delta squared term is ill-defined. The term $\delta'(x)H(x)$ is not well-defined either. △

Remark 2.1.4 (Sequential approach). One might still want an interpretation of the Leibniz rule

$$\frac{1}{2}\delta'(x) = \delta'(x)H(x) + (\delta(x))^2.$$

This is possible with so-called sequential approach of Mikusinski [27]. We interpret both terms on the righthandside together. First we regularise both terms, in a specific way dictated by the sequential approach, and then take the limit. In the regularisations the Leibniz rule can be reversed and the limit of the product corresponds to our original result $\frac{1}{2}\delta'(x)$. When manipulating expressions with several terms, a symbolic use of the Leibniz rule under the formalism of the sequential approach can be useful.

2.2 The Schwartz product

In Definition 1.3.18 we defined the Schwartz product. Although the Schwartz product is part of standard distribution theory, it does have its limitations. Most notably, the Schwartz product map is not jointly continuous. This is proven in [24]. The Schwartz product sets expectations on what is possible for any more general distributional product.

Theorem 2.2.1. *For $f \in C^\infty$ and $u \in \mathcal{D}'$, the Schwartz product fu is in $\mathcal{D}'$. The multiplication map $C^\infty \times \mathcal{D}' \to \mathcal{D}'$ is seperately continuous and jointly sequentially continuous.*

Proof. The map $\dot{f} : \mathcal{D} \to \mathcal{D} : \varphi \mapsto f\varphi$, is an endomorphism since if $\varphi_\lambda \to \varphi$ in $\mathcal{D}$, then

$$|f\varphi_\lambda - f\varphi|_{j,K} \leq |f|_{j,K}|\varphi_\lambda - \varphi|_{j,K} \to 0,$$

for all $j \in \mathbb{N}$ and $K \subset \mathbb{R}^n$ compact. Additionally there is a compact U such that

$$\operatorname{supp} f\varphi_\lambda \subseteq \operatorname{supp} \varphi_\lambda \subseteq U, \text{for all } \lambda \geq \lambda_0.$$

Thus $\dot{f}$ is a continuous linear map $\mathcal{D} \to \mathcal{D}$. Composition with the continuous functional u, implies $fu \in \mathcal{D}'$.

For seperate continuity take the net $C^\infty \supset f_\lambda \to f$ and fix $u \in \mathcal{D}'$. We have

$$|f_\lambda\varphi - f\varphi|_{j,K} \leq |f_\lambda - f|_{j,K}|\varphi|_{j,K},$$

which implies $f_\lambda\varphi \to f\varphi$ since the net $f_\lambda\varphi$ has bounded support. Due to continuity of u, we get $f_\lambda u \to fu$. Next let $\supset u_\lambda \to u$ in the strong topology of $\mathcal{D}'$. We need to verify

$$\sup_{\varphi \in B} |\langle fu_\lambda, \varphi\rangle - \langle fu, \varphi\rangle| = \sup_{\varphi \in B} |\langle u_\lambda - u, f\varphi\rangle| \to 0, \tag{2.1}$$

for all bounded sets $B \subset \mathcal{D}$. But

$$|f_\varphi|_{j,K} \leq |f|_{j,K}|\varphi|_{j,K}$$

such that $\tilde{B} = \{f\varphi, \varphi \in B\}$ is again bounded. Applying strong convergence of u_λ to u on $\tilde{B}$ in (2.1) then gives convergence of fu_λ to fu.

Sequential continuity is without proof. □

Fully analagously, one proves the same properties for the Schwartz product for tempered distributions

$$\mathcal{O}_M(\mathbb{R}^n) \times \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

The main idea in the Schwartz method is the use of the multiplier spaces $C^\infty(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n)$ respectively. The approach can be generalised to the duality method of section 2.4. Changing the duality $\mathcal{D}'(\mathbb{R}^n) - \mathcal{D}(\mathbb{R}^n)$ to general $X' - X$ leads to different multiplier spaces and therefore different products.

The Schwartz product also allows for the the Leibniz rule. Let $u \in \mathcal{D}'$, $f \in C^\infty$ and $\varphi \in \mathcal{D}$. We compute the derivative of the product directly

$$\begin{aligned}
\langle \partial_{x_i}(fu), \varphi\rangle &= -\langle fu, \partial_{x_i}\varphi\rangle \\
&= \langle u, f\partial_{x_i}\varphi\rangle \\
&= -\langle u, \partial_{x_i}(f\varphi) - \partial_{x_i}(f)\varphi\rangle \\
&= \langle \partial_{x_i}u, f\varphi\rangle + \langle \partial_{x_i}(f)u, \varphi\rangle \\
&= \langle f\partial_{x_i}u + \partial_{x_i}(f)u, \varphi\rangle.
\end{aligned}$$

Example 2.2.2. We have the Schwartz product

$$x^k\delta^{(j)} = \begin{cases} 0, & \text{if } j < k, \\ (-1)^k k!\binom{j}{k}\delta^{(j-k)}(x), & \text{if } k \leq j. \end{cases}$$

△

2.3 Localisation and product by disjoint singular support

Localisation exploits the local nature of the product of generalised functions. For the Schwartz product, we have the support property

$$\operatorname{supp} uv \subseteq \operatorname{supp} u \cap \operatorname{supp} v, \quad u \in C^\infty(\mathbb{R}^n), v \in \mathcal{D}'(\mathbb{R}^n).$$

We naturally generalise this property to hold for all $u, v \in \mathcal{D}'(\mathbb{R}^n)$.

We define the local product. To work in full generality, let X be a function space and X' be its dual space. Write $X(\Omega)$ for the functions in X with support in $\Omega \subseteq \mathbb{R}^n$ and call $X'(\Omega)$ its dual space.

Define the product of distributions $u, v \in X$ around $x \in \mathbb{R}^n$ in a open neighborhood Ω_x of x by

$$_{X(\Omega_x)'}\langle w_x, \psi \rangle_{X(\Omega_x)} := {}_{X'}\langle (f_x u) \cdot (f_x v), \psi \rangle_X,$$

where $f_x \equiv 1$ on Ω_x is any multiplier of X. It is clear that the value at x is independent of f_x and Ω_x because if $\tilde{f}_x \equiv 1$ on $\tilde{\Omega}_x$, then the w_x and $\tilde{w}_x$ agree on $\Omega_x \cap \tilde{\Omega}_x$.

Additionally we have consistency in the following way. If $y \in \Omega_x$, then (f_x, Ω_x) is also a localization around y. It remains to combine the locally defined products into one global product. If for all x the products are well-defined locally in Ω_x as above, then the sewing lemma guarantees a unique global distribution. See Treves [35].

Theorem 2.3.1 (Sewing lemma)**.** *Let $\{U_i\}_{i \in I}$ be an open cover of Ω, that is, $\Omega = \bigcup_{i \in I} U_i$. Suppose $\{f_i\}_{i \in I}$ is a family such that*

1. *$f_i \in \mathcal{D}'(U_i)$, for each $i \in I$.*
2. *If $U_i \cap U_j \neq \emptyset$, then $f_i = f_j$ on $U_i \cap U_j$.*

then there is a unique distribution $f \in \mathcal{D}'(\Omega)$ such that $f = f_i$ on U_i.

The localization procedure lets us extend the definition of the Schwartz product.

Definition 2.3.2 (Singular support)**.** For $u, v \in \mathcal{D}(\mathbb{R}^n)$, we say $u = v$ on $\Omega \subseteq \mathbb{R}^n$ iff $\langle u, \varphi \rangle = \langle v, \varphi \rangle$ for all $\varphi \in \mathcal{D}(\Omega)$. That is $u = v$ in $\mathcal{D}'(\Omega)$. Let A_u be the collection of sets on which u is smooth

$$A_u = \left\{ \Omega \subseteq \mathbb{R}^n \text{ open }; \exists f \in (\mathbb{R}^n) : f = u \text{ on } \Omega \right\}.$$

Then the singular support singsupp(u) of u is

$$\operatorname{singsupp}(u)^c = \bigcup_{\Omega \in A_u} \Omega,$$

which is the smallest closed set on which u is not smooth.

This means we can find $a \in \mathcal{D}'(\mathbb{R}^n)$ and $g \in C^\infty(\mathbb{R}^n)$ such that $u = g + a$ and $\operatorname{supp}(a) \subseteq \operatorname{singsupp}(u)$. We can now define the product by disjoint singular support as the localization of the Schwartz product.

Definition 2.3.3 (Product by disjoint singular support)**.** For $u, v \in \mathcal{D}$ with $\operatorname{singsupp}(u) \cap \operatorname{singsupp}(v) = \emptyset$, we define the product by disjoint singular support as

$$uv = fg + fb + ga.$$

with $u = f + a, v = g + b$ where $f, g \in$, $a, b \in \mathcal{D}'$, $\operatorname{supp}(a) \subseteq \operatorname{singsupp}(u)$ and $\operatorname{supp}(b) \subseteq \operatorname{singsupp}(v)$.

This product directly inherits any local properties of the Schwartz product. If we fix two open disjoint sets Ω_1, Ω_2, then the multiplication map $(C^\infty + \mathcal{D}'(\Omega_1)) \times (C^\infty + \mathcal{D}'(\Omega_2))$ has the same properties as the Schwartz product.

The localisation can be used extend the definition any product map. We will therefore stay at a global treatment of such products, knowing that a localisation is always possible. Next we discuss the duality method, which extends the Schwartz product.

2.4 Duality method

The duality method is used to define a wide class of multiplication maps. Each map is for pairs of distributions that are locally in a TVS X and its dual X'. We ensure that such multiplication is interpretable as a multiplication of distributions and that the multplication map has sufficient continuity properties.

Definition 2.4.1. We call a topological vectorspace X normal if

1. The inclusions $\mathcal{D} \subseteq X \subseteq \mathcal{D}'$ hold.
2. The space of testfunctions $\mathcal{D}$ is dense in X.

3. The inclusion map $\mathcal{D} \hookrightarrow X$ is continuous.

4. The inclusion map $X \hookrightarrow \mathcal{D}'$ is continuous.

We call X normal in the TVS Y if the above properties hold for Y instead of $\mathcal{D}$.

The dual space X' of a normal space X is neccesarily normal. We define the space of multipliers X_{loc} by

$$X_{\text{loc}} = \{u \in \mathcal{D} : u\varphi \in X, \forall \varphi \in \mathcal{D}\}.$$

The simplest example is $\mathcal{D}_{\text{loc}} = C^\infty$. Another common example is $X = L^p$ with $X_{\text{loc}} = L^p_{\text{loc}}$.

Definition 2.4.2 (Duality product)**.** Let X be a normal space. We define the duality product of $u \in (X')_{\text{loc}}$ and $v \in X_{\text{loc}}$ by

$$_{\mathcal{D}}\langle uv, \varphi\rangle_{\mathcal{D}} = {}_{X'}\langle \chi u, v\varphi\rangle_X = {}_{X'}\langle \chi u, (\chi v)\varphi\rangle_X.$$

where $\chi \in \mathcal{D}$ with $\chi = 1$ on $\operatorname{supp}(\varphi)$.

We will further write X'_{loc} for $(X')_{\text{loc}}$ and $(X_{\text{loc}})'$ if we mean the other interpretation of the symbol.

Theorem 2.4.3. *Definition 2.4.2 does not depend on the choice of χ. The resulting product is commutative and partially associative and produces a separately continuous bilinear map $X'_{\text{loc}} \times X_{\text{loc}} \to \mathcal{D}'$.*

Proof. Take $\chi, \varphi \in \mathcal{D}$ be as in the definition of the duality product. Now let $w_\varepsilon \in \mathcal{D}$ converge to $u\chi$ in $\mathcal{D}'$ for $\varepsilon \to 0$ and take $\chi_2 \in \mathcal{D}$ like χ. Then

$$_{X'}\langle w_\varepsilon, \varphi v\rangle_X = {}_{X'}\langle \psi w_\varepsilon, \varphi v\rangle_X \to {}_{X'}\langle \psi\chi u, \varphi v\rangle_X.$$

It follows that $_{X'}\langle \psi\chi u, \varphi v\rangle_X = {}_{X'}\langle \psi u, \varphi v\rangle_X = {}_{X'}\langle \chi u, \varphi v\rangle_X$.

For commutativity, take $\psi \in \mathcal{D}$ with $\psi = 1$ on $\operatorname{supp}(\varphi)$ so that

$$_{X'}\langle \chi u, \varphi v\rangle_X = {}_{X'}\langle \chi u, \varphi\psi v\rangle_X = {}_{X'}\langle \varphi\chi u, \psi v\rangle_X = {}_{X'}\langle \varphi u, \chi\psi v\rangle_X,$$

where $\chi\psi$ equals 1 on $\operatorname{supp}(\varphi)$. For associativity, let $u, v, w \in \mathcal{D}'$. Suppose $w \in X_{\text{loc}}$ and $uv \in X'_{\text{loc}}$, X normal. Suppose $vw \in Z'_{\text{loc}}$, $u \in Z_{\text{loc}}$, Z normal. Further suppose $v \in Y'_{\text{loc}}$, $u \in Y_{\text{loc}}$, with Y normal in X and in Z.

$$\begin{aligned} _{\mathcal{D}'}\langle (uv)w, \varphi\rangle_{\mathcal{D}} &= {}_{X'}\langle \chi_1 uv, \varphi w\rangle_X = {}_{Y'}\langle \chi_1\chi_2 v, \varphi uw\rangle_Y \\ &= {}_{Z'}\langle \chi_1 vw, \varphi u\rangle_Z = {}_{\mathcal{D}'}\langle u(vw), \varphi\rangle_{\mathcal{D}}. \end{aligned}$$

The seperate continuity of the multiplication map follows by composition from the continuity of the action in X and the continuity of the multiplication maps $X_{\text{loc}} \times \mathcal{D} \to X$ and $X'_{\text{loc}} \times \mathcal{D} \to X$. ☐

Example 2.4.4. The Schwartz product $C^\infty \times \mathcal{D}' \to \mathcal{D}'$ follows by taking $X = \mathcal{D}$. We have $\mathcal{D}_{\text{loc}} = C^\infty$ and $\mathcal{D}'_{\text{loc}} = \mathcal{D}'$. Taking $X = \mathcal{S}$ or $X = \mathcal{E}$ results in the same product. △

Example 2.4.5. Let $X = \mathcal{D}^N(\mathbb{R}^n)$, the functions $\mathbb{R}^n \to \mathbb{C}$ of compact support that are N times continuously differentiable. Its dual space is $\mathcal{D}'^N(\mathbb{R}^n)$, the distributions of order N. These are the elements $u \in \mathcal{D}'(\mathbb{R}^n)$ that satisfy

$$|_{\mathcal{D}'}\langle u, \varphi\rangle_{\mathcal{D}}| \leq M_K ||\varphi||_N,$$

for all $\varphi \in \mathcal{D}$ with $\operatorname{supp}\varphi \subseteq K$, for some positive constant M_K. We have $X_{\text{loc}} = C^N$, the N times continuously differentiable functions, and $X'_{\text{loc}} = \mathcal{D}'^{(m)}$. The duality method thus defines the multiplication $C^N \times \mathcal{D}'^N \to \mathcal{D}'$. △

Example 2.4.6. For $1 \leq p, q \leq \infty$ the spaces L^p and L^q are dual when $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Their multiplier spaces are L^p_{loc} and L^q_{loc} respectively. The duality method then defines the multiplication map $L^p_{\text{loc}} \times L^q_{\text{loc}} \to \mathcal{D}'$. By Hölder's inquality the pointwise product $L^p_{\text{loc}} \times L^q_{\text{loc}}$ is in $L^1_{\text{loc}} \subseteq \mathcal{D}'$. △

Example 2.4.7 (Sobolev product)**.** For $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\ell \geq 0$, the Sobolev space $W^{\ell,p}$ is normal and we have the duality $(W^{-\ell,p} - W^{\ell,q})$, $\ell \geq 0$. We can thus apply the duality method to $X = W^{\ell,p}$ resulting in a multiplication map $W^{-\ell,p}_{\text{loc}} \times W^{\ell,q}_{\text{loc}} \to \mathcal{D}'$. △

Theorem 2.4.8. *Let ℓ, $m \in \mathbb{Z}$, $1 \leq p, q \leq \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$ and $\ell + m \geq 0$. Let $k = \min(\ell, m)$, then the duality method defines a multiplication map $W^{m,q}_{\text{loc}} \times W^{\ell,p}_{\text{loc}} \to W^{k,r}_{\text{loc}}$.*

Proof. We have $W^{m,q}_{\text{loc}} \subseteq (W^{\ell,p})'_{\text{loc}}$. Then for $u \in W^{m,q}_{\text{loc}}$ and $v \in W^{\ell,p}_{\text{loc}}$ the duality method produces the multiplication map defined by

$$\langle uv, \varphi \rangle = {}_{W^{m,q}}\langle \chi u, (\chi v)\varphi \rangle_{W^{\ell,p}},$$

where $\chi(x) = 1$ on $\text{supp}(\varphi)$. Given the parameters as in the statement of the theorem, we verify that $uv \in \mathcal{D}'$ coincides with an element of $W^{k,r}_{\text{loc}}$. Working locally, we consider $\varphi \in \mathcal{D}(\omega)$ with ω an open and bounded subset of $\mathbb{R}^n$.

In the case when $\ell, m \geq 0$, we simply have

$$\langle uv, \varphi \rangle = {}_{W^{m,q}}\langle \chi u, (\chi v)\varphi \rangle_{W^{\ell,p}} = \int_\omega (\chi u)(\chi v)\varphi,$$

where the pointwise multiplication $(\chi u)(\chi v)$ is in $L^r(\omega)$, such that the integrandum is integrable.

In the case when $k \geq 0$, we notice that additionally that $\partial^\alpha(\chi u) \in L^p(\omega)$ resp. $\partial^\alpha(\chi v) \in L^q(\omega)$ for any multi-index $\alpha \in \mathbb{N}^n, |\alpha| \leq k$. By the Leibniz rule for positive order Sobolev spaces

$$\partial^\alpha(\chi u\, \chi v) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}(\chi u)\partial^\beta(\chi v),$$

in which the righthandside is clearly $L^r(\omega)$ as each term is has factors in $L^p(\omega)$ and $L^q(\omega)$. Next we have the case where $k = m < 0$ but $\ell + m \geq 0$. We have the representation $u = \sum_{|\alpha| \leq m} \partial^\alpha u_\alpha$ for $u_\alpha \in L^q_{loc}$. On ω we get

$$\begin{aligned}
\langle uv, \varphi \rangle &= \sum_{|\alpha| \leq m} {}_{W^{m,q}}\langle \partial^\alpha(\chi u_\alpha), \chi v \varphi \rangle_{W^{\ell,p}} \\
&= \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \int_\omega \chi u_\alpha \partial^\alpha(\chi v \varphi) \\
&= \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_\omega \chi u_\alpha \partial^{\alpha-\beta}(\chi v)\partial^\beta \varphi.
\end{aligned}$$

This formula shows the action

$$\langle uv, \varphi \rangle = \langle \sum_{|\alpha| \leq m} \sum_{\beta \leq \alpha} (-1)^\beta \binom{\alpha}{\beta} \partial^\beta \left(\chi u_\alpha \partial^{\alpha-\beta}(\chi v) \right), \varphi \rangle.$$

The distribution on the righthandside is in $W^{k,r}$ since every product $\chi u_\alpha \partial^{\alpha-\beta}(\chi v)$ belongs to $L^r(\omega)$. Finally we consider the seperate continuity of the multiplication map. But this follows easily since one can use convergence in the Sobolev norms and the fact that the Sobolev spaces are continuously embedded into $\mathcal{D}'$. □

We give an elaborate example of multiplication by Sobolev spaces which shows the strenght of the duality product.

Definition 2.4.9 (Pf r^λ)**.** For $\text{Re}(\lambda) > -n$, we have that $|x|^\lambda \in L^1_{loc}(\mathbb{R}^n)$ is a regular distribution of polynomial growth as $|x| \to \infty$, thus

$$\langle \text{Pf}\, r^\lambda, \varphi \rangle = {}_{\mathcal{S}'}\langle r^\lambda, \varphi \rangle_{\mathcal{S}} = \int_{\mathbb{R}^n} |x|^\lambda \varphi dx. \tag{2.2}$$

For $\text{Re}(\lambda) > -n-2, \lambda \neq -n$ we define

$$\begin{aligned}
\langle \text{Pf}\, r^\lambda, \varphi \rangle = &\int_1^\infty r^{\lambda+n-1} \int_{S^{n-1}} \varphi(r\omega) d\omega dr \\
&+ \int_0^1 r^{\lambda+n-1} \int_{S^{n-1}} (\varphi(r\omega) - \varphi(0)) d\omega dr \\
&+ \frac{1}{\lambda+n} |S^{n-1}| \varphi(0),
\end{aligned}$$

which coincides with (2.2) for $\text{Re}(\lambda) > -n$.

To get to a general analytic extension we apply the Taylor appoximation

$$\varphi(x)=\sum_{|\alpha|\le j-1}\frac{(\partial^\alpha\varphi)(0)}{\alpha!}x^\alpha+\sum_{|\alpha|=j}R_\alpha(x)x^\alpha,$$

for which we have the bounds

$$|R_\alpha(x)|\le\frac{1}{\alpha!}\max_{|\beta|=|\alpha|}\max_{y\in B_n(0,1)}\left|\partial^\beta\varphi(y)\right|=c_\alpha.$$

Setting $x=r\omega$ we simplify the main terms

$$\int_{S^{n-1}}\frac{\partial^\alpha\varphi(0)}{\alpha!}r^{|\alpha|}\omega^\alpha d\omega=d_\alpha\frac{\partial^\alpha\varphi(0)}{\alpha!}r^{|\alpha|},$$

with

$$d_\alpha=\begin{cases}0, & \text{if any } \alpha_i \text{ is odd,}\\ 2\dfrac{\prod_{i=1}^n\Gamma\left((\alpha_i+1)/2\right)}{\Gamma\left(\dfrac{1}{2}(|\alpha|+n)\right)}, & \text{if all } \alpha_i \text{ are even.}\end{cases}$$

The coefficients are the integrals of the monomials r^α over the $n-1$-sphere (see Folland [17]).

Starting from formula (2.2) and substracting the main terms with $j=2k$ gives for $\mathrm{Re}(\lambda)>-n-2k$, $\lambda\notin\{-n,-n-2,\ldots,-n-2k\}$ the formula

$$\langle\mathrm{Pf}\,r^{\lambda+n-1},\varphi\rangle=\int_1^\infty r^{\lambda+n-1}\int_{S^{n-1}}\varphi(r\omega)d\omega dr \tag{2.3}$$

$$+\int_0^1 r^{\lambda+n-1}\int_{S^{n-1}}\left(\varphi(r\omega)-\sum_{|\alpha|\le 2k-1}d_\alpha\frac{\partial^\alpha\varphi(0)}{\alpha!}r^{|\alpha|}\right)d\omega dr \tag{2.4}$$

$$+\sum_{|\alpha|\le 2k-1}d_\alpha\frac{\partial^\alpha\varphi(0)}{\alpha!(\lambda+n+|\alpha|)}, \tag{2.5}$$

where (2.4) is now integrable over r because

$$\left|\int_{S^{n-1}}R_\alpha(r\omega)r^{|\alpha|}\omega^\alpha d\omega\right|\le c_\alpha r^{|\alpha|}\int_{S^{n-1}}|\omega^\alpha|d\omega=O(r^{|\alpha|}).$$

Notice that the terms with $|\alpha|$ odd vanish since $d_\alpha=0$. The formulae agree with (2.2) and with eachother for λ's for which each are well-defined. These formulae together define a meromorphic extension $\mathbb{C}\to\mathcal{S}'(\mathbb{R}^n):\lambda\mapsto\mathrm{Pf}(r^\lambda)$ of (2.2) with poles at $\lambda=-n,-n-2,\ldots\}$. Additionally when $\lambda=-n-2j, j\in\mathbb{N}$, we set

$$\langle\mathrm{Pf}\,r^{-n-2j},\varphi\rangle=\int_1^\infty r^{-n-2j}\int_{S^{n-1}}\varphi(r\omega)d\omega dr$$

$$+\int_0^1 r^{-n-2j}\int_{S^{n-1}}\left(\varphi(r\omega)-\sum_{|\alpha|\le j-1}d_\alpha\frac{\partial^\alpha\varphi(0)}{(\alpha)!}r^{|\alpha|}\right)d\omega dr$$

$$+\sum_{|\alpha|\le j-1}d_\alpha\frac{\partial^\alpha\varphi(0)}{\alpha!(\lambda+n+|\alpha|)},$$

where we ignore the terms in (2.5) which make the pole at $\lambda=-n-2j$. This is the so-called Hamard finite part definition for pseudofunctions.

Theorem 2.4.10. *For* $\mathrm{Re}(\lambda+\mu)>-n$, *there is* $k\in\mathbb{Z},\frac{1}{p}+\frac{1}{q}=1$ *such that* $\mathrm{Pf}(r^\lambda)\in W_{loc}^{-2k,q}$ *and* $\mathrm{Pf}(r^\mu)\in W_{loc}^{2k,q}$. *We then have the duality product formula*

$$\mathrm{Pf}\,r^\lambda\,\mathrm{Pf}\,r^\mu=\mathrm{Pf}\,r^{\lambda+\mu}.$$

Proof. For $z \in \mathbb{C}$ we have $r^z \in W^{2k,p}_{loc}$ iff $(\operatorname{Re} z - 2k)p > -n$ iff $\operatorname{Re} z > \dfrac{-n}{p} + 2k$.

Case 1: $\operatorname{Re}\lambda < 0, \operatorname{Re}\mu > -n, \operatorname{Re}\lambda \notin \{-n-2j, j \in \mathbb{N}\}$. We choose $k \in \mathbb{N}$, minimal such that $\operatorname{Re}\lambda + 2k > -n$ and let $\varepsilon > 0$ satisfie $\operatorname{Re}(\lambda+\mu) = -n+\varepsilon$. Next we find $q > 1$ with $\operatorname{Re}\lambda + 2k = \dfrac{-n}{q}$. Consequently $\operatorname{Re}\mu - 2k = \dfrac{n}{q} - n + \varepsilon = -\dfrac{n}{p} + \varepsilon$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For $\delta > 0$ small enough and $\tilde{q} = q - \delta$ we get

$$\operatorname{Re}\lambda + 2k > \frac{-n}{\tilde{q}}, \quad \operatorname{Re}\mu - 2k > \frac{-n}{\tilde{p}}.$$

Therefore $\operatorname{Pf} r^\lambda \in W^{-2k,\tilde{q}}_{loc}$ and $\operatorname{Pf} r^\mu \in W^{2k,\tilde{p}}_{loc}$. We conclude that the duality product exists with values in $L^1(\mathbb{R}^n)$. For simplicity of notation we continue by writing $\tilde{p} = p$ and $\tilde{q} = q$.

The value of this product follows from analytic continuation in μ. We first calculate its value in the halfplane $\operatorname{Re}\mu > 4k - \dfrac{n}{p}$. By the previous paragraph, the product $\operatorname{Pf} r^\lambda r^{\mu-2k}$ exists. By partial associativity with $r^{2k} \in C^\infty$ we get

$$\left(r^{2k} \operatorname{Pf} r^\lambda\right) \cdot r^\mu - 2k = \operatorname{Pf} r^\lambda \cdot \left(r^{2k} r^{\mu-2k}\right) = \operatorname{Pf} r^\lambda \cdot r^\mu.$$

On the lefthandside we have $r^{2k} \cdot \operatorname{Pf} r^\lambda = r^{\lambda+2k}$ since the expression is analytic in λ and the formula holds for $\operatorname{Re}\lambda > 0$. The lefthandside then reads $r^{\lambda+2k} \cdot r^{\mu-2k} = r^{\lambda+\mu}$, with the multiplication in the $L^p_{loc} \times L^q_{loc}$ sense. We conclude the formula

$$\operatorname{Pf}(r^\lambda) r^\mu = r^{\lambda+\mu} \tag{2.6}$$

for $\operatorname{Re}\mu > 4k - \dfrac{n}{p}$. Now for $\mu > 2k - \dfrac{n}{p}$, the map $\mu \to \operatorname{Pf} r^\mu$ is analytic with values in $W^{2k,p}_{loc}$. For fixed λ, the lefthandside of (2.6) is analytic in μ with values in $\mathcal{S}'$ due to the continuity of the duality product $W^{-2k,q}_{loc} \times W^{2k,p}_{loc}$ which applies here. By the identity theorem for holomorphic functions, (2.6) holds for $\mu > 2k - \dfrac{n}{p}$.

Case 2: $\operatorname{Re}\lambda \in \{-n-2j, j \in \mathbb{N}\}$. We have the formulae

$$\operatorname{Pf} r^{-n} = \frac{1}{2-n}\Delta\left(r^{2-n}\log r - \frac{1}{2-n} r^{2-n}\right), \quad n \geq 3,$$

$$\operatorname{Pf} r^{-2} = \frac{1}{2}\Delta \log^2 r, \quad n = 2,$$

and

$$\delta = \frac{1}{(2-n)|S^{n-1}|}\Delta r^{2-n}, \quad n \geq 3,$$

$$\delta = \frac{1}{2\pi}\Delta \log r, \quad n = 2.$$

For $n \geq 3$ we have $r^{2-n} \in L^q_{loc}$ when $q > \dfrac{n}{n-2}$, so $\operatorname{Pf} r^{-n}, \delta \in W^{-2,q}_{loc}$.
For $n = 2$, we have $\operatorname{Pf} r^{-2}, \delta \in W^{-2,q}_{loc}$ for $1 \leq q < \infty$. Next, there are constants $a_{n,k}, b_{n,k}$ such that

$$\operatorname{Pf} r^{-n-2k} = a_{k,k}\Delta^k \operatorname{Pf} r^{-n} + b_{n,k}\Delta^k \delta,$$

and thus $\operatorname{Pf} r^{-n-2k} \in W^{-2-2k,q}_{loc}$ when q is as specified before.

Now let $\lambda = -n-2k$, then neccesarily $\operatorname{Re}\mu > 2k$. We have $r^\mu \in W^{2+2k,p}_{loc}$ when

$$\operatorname{Re}\mu = 2k + \varepsilon_1 > \frac{-n}{p} + 2k + 2 \iff \frac{n}{p} > \varepsilon_1 \iff 1 \leq p \leq \frac{n}{2} + \varepsilon_2.$$

This agrees with our choice of q, by $\dfrac{1}{p} + \dfrac{1}{q} = 1$. We conclude that the duality product $\operatorname{Pf} r^\lambda \cdot \operatorname{Pf} r^\mu$ exists in case 2.

To compute the value of the product like in case 1 by considering partial associativity with $r^{2k+2} \in C^\infty$.

$$\left(r^{2k+2} \operatorname{Pf} r^\lambda\right) \cdot r^{\mu-2k-2} = \operatorname{Pf} r^\lambda \cdot \left(r^{2k+2} r^{\mu-2k+2}\right) = \operatorname{Pf} r^\lambda \cdot r^\mu.$$

On the lefthandside we find $r^{2k+2} \cdot \operatorname{Pf} r^\lambda = F.p.r^{2k+2} r^\lambda = r^{\lambda+2k+2}$, since the finite part and the multiplication by r^{2k+2} can be interchanged.

Case 3: $\operatorname{Re}\lambda \in \{-n, -n-2, \dots\}, \operatorname{Im}\lambda \neq 0$. By the same duality of case 2. The value is calculated without need to consider finite parts. $\square$

The multiplication map for many choices of the normal space X can be extended.

2.5 Fourier product

The Fourier product exploits a generalisation of the exchange formula (1.7). To ensure good properties of the resulting multplication, the definition of convolution needs to be restricted to the $\mathcal{S}'$-convolution.

For $S, T \in L^1(\mathbb{R}^n)$ we have that $S * T \in L^1 \subseteq \mathcal{S}'(\mathbb{R}^n)$. Interpreting this as the convolution of two regular tempered distributions allows the following manipulation:

$$\begin{aligned} {}_{\mathcal{S}'(\mathbb{R}^n)}\langle S * T, \varphi\rangle_{\mathcal{S}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} S(y)T(x-y)\varphi(x)dydx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} S(-y)\varphi(x-y)T(x)dydx \\ &= {}_{L^1(\mathbb{R}^n)}\langle(\varphi * \check{S})T, 1\rangle_{L^\infty(\mathbb{R}^n)}, \end{aligned}$$

where $\check{S}(x) = S(-x)$. The $\mathcal{S}'$-convolution changes the duality of this action, allowing only certain pairs (S, T) to have an interpretable convolution. We try to find a distribution space with low restrictions for $(\varphi * \check{S})T$ with 1 in it's dual, while maintaining properties of the convolution.

Definition 2.5.1. The space of integrable distributions is defined as

$$\mathcal{D}'_{L^1}(\mathbb{R}^n) = \bigcup_{m\geq 0} W^{-m,1}(\mathbb{R}^n),$$

and its dual space is the space of smooth functions with bounded derivatives

$$\mathcal{D}_{L^\infty}(\mathbb{R}^n) = \bigcap_{m\geq 0} W^{m,\infty}(\mathbb{R}^n).$$

This duality allows the definition of the $\mathcal{S}'$-convolution.

Definition 2.5.2. Let $S, T \in \mathcal{S}'(\mathbb{R}^n)$. If for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have that $(\varphi * \check{S})T \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$, then the $\mathcal{S}'$-convolution of S and T exists and is defined by

$${}_{\mathcal{S}'(\mathbb{R}^n)}\langle S * T, \varphi\rangle_{\mathcal{S}(\mathbb{R}^n)} = {}_{\mathcal{D}'_{L^1}(\mathbb{R}^n)}\langle(\varphi * \check{S})T, 1\rangle_{\mathcal{D}_{L^\infty}(\mathbb{R}^n)}.$$

To make this definition more practical, one can find pairs of subspaces (X, Y) of $\mathcal{S}'(\mathbb{R}^n)$ such that the convolution exists for all $S \in X$ and $T \in Y$.

Theorem 2.5.3. *Let $S, T \in \mathcal{S}'(\mathbb{R}^n)$ and suppose that $(\varphi * \check{S})T \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then the map*

$$\mathcal{S}(\mathbb{R}^n) \to \mathbb{C} : \varphi \mapsto {}_{\mathcal{D}'_{L^1}(\mathbb{R}^n)}\langle(\varphi * \check{S})T, 1\rangle_{\mathcal{D}_{L^\infty}(\mathbb{R}^n)},$$

is continuous. Thus

$${}_{\mathcal{S}'(\mathbb{R}^n)}\langle S * T, \varphi\rangle_{\mathcal{S}(\mathbb{R}^n)} = {}_{\mathcal{D}_{L^1}(\mathbb{R}^n)}\langle(\varphi * \check{S})T, 1\rangle_{\mathcal{D}_{L^\infty}(\mathbb{R}^n)}.$$

Proof. The topology on $\mathcal{D}'_{L^1}$ is induced by the seminorms $|\cdot|_{W^{-m,1}}$, $m \geq 1$. The map $\mathcal{S} \to \mathcal{O}_M : \varphi \mapsto \varphi * \check{S}$ is continuous. The continuity of the multiplication $\mathcal{O}_M \cdot \mathcal{S}'$ implies that $\mathcal{S} \to \mathcal{S}' : \varphi \mapsto (\varphi * \check{S})T$ is continuous as well. Restricting this map to its range $\mathcal{D}'_{L^1}$ keeps continuity. Then evaluating at 1 gives the continuous linear functional $\mathcal{S} \to \mathbb{C} : \varphi \mapsto \langle(\varphi * \check{S})T, 1\rangle$. $\square$

We are interested in the product on the lefthandside. Applying the inverse Fourier transform gives

$$fg = \mathcal{F}^{-1}(\mathcal{F}(f) * \mathcal{F}(g)).$$

We can interpret this relation for general $f, g \in \mathcal{S}'$ provided that the righthandside makes sense. Formally we define the Fourier product.

Definition 2.5.4. Let $S, T \in \mathcal{S}'$ such that the S'-convolution $U = \mathcal{F}S * \mathcal{F}T$ exists. Then the Fourier product of S and T is

$$ST = \mathcal{F}^{-1}(\mathcal{F}(S) * \mathcal{F}(T)).$$

The Fourier product can be improved using localization. For $x \in \mathbb{R}^n$ let $\chi_x \in \mathcal{D}(\mathbb{R}^n)$ equal one on a neighborhood $\Omega_x \subset \mathbb{R}^n$ of x. Suppose that the Fourier product of $\chi_x S$ and $\chi_x T$ in $\mathcal{D}'(\Omega_x)$ exists for every $x \in \mathbb{R}^n$, then the sewing lemma defines a global distribution.

The approach by a generalized exchange formula (1.7) is due to Vladimirov (see [36]). His method was then localized by Oberguggenberger in the article 'Products of distributions' [29]. Another product of distributions based on the Fourier transform was given by Ambrose in [6]. Oberguggenberger proved in [29] that the two definitions are equivalent.

Example 2.5.5. The Fourier product of $S = 1/(x + i0)$ and $T = 1/(x + i0)$ exists. First we have $\mathcal{F}S = \mathcal{F}T = \mathcal{F}\dfrac{1}{x + i0} = -2\pi i H(-x)$ by the reflection formula of the Fourier transform. The convolution $(-2\pi i\check{H}) * (-2\pi i\check{H})(x)$ is calculated by

$$\begin{aligned}
\check{H} * \check{H}(x) &= \int_{\mathbb{R}} H(-\xi)H(\xi - x)d\xi \\
&= \begin{cases} \int_x^0 d\xi = -x, & \text{if } x \leq 0, \\ 0, & \text{if } x \geq 0, \end{cases} \\
&= -xH(-x).
\end{aligned}$$

For the well-definedness of $\mathcal{S}'$-convolution we check for $\varphi \in \mathcal{S}(\mathbb{R})$ that $(\varphi * H)\check{H} \in \mathcal{D}'_{L^1}(\mathbb{R})$. But we have

$$(\varphi * H)(x) = \int_{\mathbb{R}} \varphi(\xi)H(x - \xi)d\xi = \int_{-\infty}^{x} \varphi(\xi)d\xi,$$

and therefore

$$(\varphi * H)(x)\check{H}(x) = H(-x)\int_{-\infty}^{x} \varphi(\xi)d\xi \in L^1(\mathbb{R}) \subset \mathcal{D}'_{L^1}(\mathbb{R}).$$

It remains to calculate that

$$\mathcal{F}^{-1}(4\pi^2 xH(-x)) = \mathrm{Pf}(\frac{1}{x^2}) + \pi i\delta'(x).$$

First we notice that

$$-2\pi i\mathcal{F}(xf(x)) = \frac{d}{dx}\mathcal{F}(f(x)),$$

for any $f \in \mathcal{S}'(\mathbb{R})$. Thus

$$\begin{aligned}
\mathcal{F}^{-1}(4\pi^2 xH(-x)) &= \frac{4\pi^2}{2\pi i}\frac{d}{dx}\mathcal{F}\check{H} = -2\pi i\frac{d}{dx}\frac{1}{2\pi i}\frac{1}{x - i0} \\
&= -\frac{d}{dx}\left(\text{v.p.}\frac{1}{x} + \pi i\delta(x)\right) = \mathrm{Pf}\,\frac{1}{x^2} - \pi i\delta'(x).
\end{aligned}$$

△

Example 2.5.6. The Fourier product of $S = \dfrac{1}{x + i0}$ and $T = \dfrac{1}{x - i0}$ does not exist. We have $T(x) = -S(-x)$, thus also

$$\mathcal{F}(T)(x) = -\mathcal{F}(S)(-x) = 2\pi iH(x)$$

. Now the $\mathcal{S}'$-convolution of $\mathcal{F}S$ and $\mathcal{F}T$ is not defined. For $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$R_\varphi(x) = (\varphi * \check{S})(x)T(x) = 4\pi^2 H(x)\int_{-\infty}^{x} \varphi(\xi)d\xi. \tag{2.7}$$

It is clear that R_φ is constant for $x \geq \max(0, \sup\{x \in \operatorname{supp}\varphi\})$. This implies that R_φ is not in any $W^{-m,1}(\mathbb{R})$, for $m \geq 0$. Therefore it is not a $\mathcal{D}'_{L^1}(\mathbb{R})$ distribution. We conclude that the $\mathcal{S}'$-convolution of the Fourier transforms of S and T do not exist and therefore neither does the Fourier product. △

2.6 Strict product

The strict product is defined trough regularisation and passage to the limit. The strict product employs the class of strict delta nets. We recall the definition of strict delta nets 1.3.36. A strict delta net is a net of test functions $(\rho^\varepsilon)_{\varepsilon\in(0,1]} \subset \mathcal{D}(\mathbb{R}^n)$ with the properties

$$\operatorname{supp}(\rho^\varepsilon) \to \{0\}, \quad \text{when } \varepsilon \to 0,$$
$$\int_{\mathbb{R}^n} \rho^\varepsilon(x)dx = 1, \quad \text{for all } \varepsilon \in (0,1],$$
$$\int_{\mathbb{R}^n} |\rho^\varepsilon(x)|dx \text{ is uniformly bounded for } \varepsilon \in (0,1].$$

We want to define the product of $u, v \in \mathcal{D}'(\mathbb{R}^n)$. The approach is as follows. Let $(\rho^\varepsilon)_{\varepsilon\in(0,1]}$ be a strict delta net. The regularisations $u^\varepsilon = u * \rho^\varepsilon$ and $v^\varepsilon = v * \rho^\varepsilon$ are smooth for each $\varepsilon > 0$. Using the Schwartz product, we can approximate the product uv by any of the products $u^\varepsilon v$, uv^ε or $u^\varepsilon v^\varepsilon$. It remains to see that the products have a limit as $\varepsilon \to 0$. To ensure nice properties of the resulting product, we ask that the limit exists and is equal for all strict delta nets ρ^ε. Formally we define strict products like this.

Definition 2.6.1 (Strict product)**.** We define four different strict products. Notation depends on which term is mollified. Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$, then

$$u[v] = \lim_{\varepsilon\to 0} u(\rho^\varepsilon * v), \tag{strict1}$$
$$[u]v = \lim_{\varepsilon\to 0} (\rho^\varepsilon * u)v, \tag{strict2}$$
$$[u][v] = \lim_{\varepsilon\to 0} (\rho^\varepsilon * u)(\sigma^\varepsilon * v) \tag{strict3}$$
$$[uv] = \lim_{\varepsilon\to 0} (\rho^\varepsilon * u)(\rho^\varepsilon * v), . \tag{strict4}$$

The product exists when the limit exists in $\mathcal{D}'(\mathbb{R}^n)$ and is the same for all strict delta nets ρ^ε and σ^ε.

Strict products (strict1) - (strict3) are equivalent, while (strict4) is more general. The first is proven in Theorem 2.6.2, which also gives a characterisation of the products that is useful to the strict product. A corresponding characterisation for product strict4 is given in Theorem 2.6.3.

Theorem 2.6.2. *For any two of distributions $u, v \in \mathcal{D}(\mathbb{R}^n)$, strict products* (strict1), (strict2) *and* (strict3) *are equivalent to the condition:*

$$\begin{gathered}\textit{For all } \varphi \in \mathcal{D}(\mathbb{R}^n) \textit{ there is a neighborhood } U \textit{ of zero} \\ \textit{so that } (\varphi u) * \check{v} \textit{ belongs to } L^\infty(U) \textit{ and is continuous at zero.}\end{gathered} \tag{strict5}$$

Strict products (strict1)-(strict3) *then satisfy*

$$\langle u[v], \varphi\rangle = \langle [u]v, \varphi\rangle = \langle [u][v], \varphi\rangle = (\varphi u) * \check{v}(0).$$

Proof. We first show that strict products (strict1)-(strict3) are equivalent.

(strict3) $\Longrightarrow$ (strict1) and (strict2). It's sufficient to prove that the double limit $\lim_{\varepsilon\to 0,\eta\to 0} (\rho^\varepsilon * u)(\sigma^\eta * v)$ exists and equals $[u][v]$, since one can freely take one of the variables to its limit.

The limit is equal to $[u][v]$ if

$$\lim_{k\to\infty} (\rho^{\varepsilon(k)} * u)(\sigma^{\eta(k)} * v) = [u][v], \tag{2.8}$$

for all delta sequences $\rho^{\varepsilon(k)}$ and $\sigma^{\eta(k)}$. So take sequences $\rho^{\varepsilon(k)}$ and $\sigma^{\eta(k)}$ which satisfy definition 1.3.36 for $\varepsilon(k) \to 0$ resp. $\eta(k) \to 0$ when $k \to \infty$. We extend the delta sequences to delta nets by setting

$$\tilde{\rho}^\varepsilon = \rho^{\varepsilon(k)},$$

for $\frac{1}{k+1} \leq \varepsilon < \frac{1}{k}$, and $\tilde{\sigma}^\eta$ analagously. Condition strict3 then implies that

$$\lim_{\varepsilon\to 0,\eta\to 0} (\tilde{\rho}^\varepsilon * u)(\tilde{\sigma}^\eta * v),$$

equals $[u][v]$. It is then clear that 2.8 must hold, proving the assertion.

(strict1) $\implies$ (strict3). By localization, we can assume that u and v have compact support. With some rescaling assume that $\text{supp}(u), \text{supp}(v), \text{supp}(\varphi) \subset Q = [-\pi, \pi]^n$. This allows us to use the Fourier series

$$\varphi(x) = \sum_{m \in \mathcal{Z}\setminus} c_m e^{im\cdot x},$$

for coefficients $c_m \in \mathbb{C}$ which satisfie

$$\sum_{m \in \mathcal{Z}\setminus} |c_m|(1+|m|)^k \infty.$$

Absolute convergence of the series expansion implies the convergence of the series to φ in $C^\infty(Q)$. The following change of variables will allow us to apply strict product (strict1).

$$\begin{aligned}
&\langle (\rho^\varepsilon * u)(\sigma^\varepsilon * v), \varphi \rangle, \\
&= \sum_{m \in \mathbb{Z}^n} \int\int\int u(x-y)\rho^\varepsilon(y) v(z)\sigma^\varepsilon(x-z) e^{im\cdot x} dz dy dx, \\
&= \sum_{m \in \mathbb{Z}^n} \int\int\int u(x) v(z) \rho^\varepsilon(-y)\sigma^\varepsilon(x-y-z) e^{-im\cdot y} e^{im\cdot x} dy dz dx, \\
&= \sum_{m \in \mathbb{Z}^n} \langle u(v * (\check{\rho}^\varepsilon e^{-im\cdot y} * \sigma^\varepsilon)), e^{im\cdot x} \rangle,
\end{aligned}$$

where the integral notation is only formal as to show the explicit change of variables. Call $\tau_m^\varepsilon = \check{\rho}^\varepsilon e^{-im\cdot y} * \sigma^\varepsilon$. For fixed m, τ_m^ε is almost a strict delta net. We verify the properties.

$$\text{supp}(\tau_m^\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0,$$

since $\text{supp}(\check{\rho}^\varepsilon e^{-im\cdot y}) \to 0$ and $\text{supp}(\sigma^\varepsilon) \to 0$. Secondly,

$$\int |\tau_m^\varepsilon| \leq \int |\rho^\varepsilon(x)| dx \int |\sigma^\varepsilon(y)| dy,$$

is bounded independently of ε. The last property is slightly different.

$$\left| \int \tau_m^\varepsilon(x) dx - 1 \right| \leq \left| \int \rho^\varepsilon(-y)\sigma^\varepsilon(x-y)(e^{-im\cdot y} - 1) \right|, \tag{2.9}$$

$$\leq C \sup_{y \in \text{supp}(\rho^\varepsilon)} \left| e^{-im\cdot y} - 1 \right|, \tag{2.10}$$

where $C = \int |\sigma^\varepsilon(x)| dx$. It is clear that (2.10) goes to zero such that

$$c_{\tau_m}(\varepsilon) = \int \tau_m^\varepsilon(x) dx \to 1. \tag{2.11}$$

We create a delta net by setting $\tilde{\tau}_m^\varepsilon = \tau_m^\varepsilon / c_{\tau_m}(\varepsilon)$. It is then clear that in $\mathcal{D}'(\mathbb{R}^n)$

$$\lim_{\varepsilon \to 0} u(v * \tau_m^\varepsilon) = \lim_{\varepsilon \to 0} u(v * \tilde{\tau}_m^\varepsilon) = u[v].$$

Due to the compact support of u and v, we also have convergence in $\mathcal{E}'(\mathbb{R}^n)$. All that remains is changing the sum and the limit. Since the net $u(v * \tau_m^\varepsilon)$ is convergent in $\mathcal{E}'(\mathbb{R}^n)$, we get for ε small enough

$$\langle u(v * \tau_m^\varepsilon), \psi \rangle \leq C \sup_{|\alpha| \leq k} \sup_{x \in Q} |D^\alpha \psi(x)|,$$

uniformly for $\psi \in \mathcal{E}$ for some $C > 0$ and $k \in \mathbb{Z}$. Restricting to $e^{im\cdot x}$ for $m \in \mathbb{Z}^n$, gives

$$\langle u(v * \tau_m^\varepsilon), e^{im\cdot x} \rangle \leq C(1+|m|)^k.$$

Therefore we can switch series and limit

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \langle (\rho^\varepsilon * u)(\sigma^\varepsilon * v), \varphi \rangle \\
&= \sum_{m \in \mathbb{Z}} c_m \lim_{\varepsilon \to 0} \langle u(v * \tau_m^\varepsilon), e^{im\cdot x} \rangle \\
&= \sum_{m \in \mathbb{Z}} c_m \langle u[v], e^{im\cdot x} \rangle \\
&= \langle u[v], \varphi \rangle.
\end{aligned}$$

(strict5) $\implies$ (strict1). The property follows directly from the simple observation that by definition of the convolution

$$\langle u(v * \rho^\varepsilon), \varphi\rangle = \langle \varphi u, v * \rho^\varepsilon\rangle = \langle (\varphi u) * \check{v}, \rho^\varepsilon\rangle.$$

Now suppose (strict5) holds, then

$$\begin{aligned}\langle (\varphi u) * \check{v}, \rho^\varepsilon\rangle - (\varphi u) * \check{v}(0) &= \int (((\varphi u) * \check{v})(x) - ((\varphi u) * \check{v})(0))\rho^\varepsilon(x)dx \\ &\leq \sup_{x \in \operatorname{supp}(\rho^\varepsilon)} |((\varphi u) * \check{v})(x) - ((\varphi u) * \check{v})(0)| \int |\rho^\varepsilon(x)|dx \to 0.\end{aligned}$$

(strict1) $\implies$ (strict5). Fix $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and assume by (strict1) that

$$c = \lim_{\varepsilon \to 0} \langle (\varphi u) * \check{v}, \rho^\varepsilon\rangle. \tag{2.12}$$

The existence of (2.12) for all delta nets ρ^ε draws us to interpret them as a duality, where $\rho^\varepsilon \to \delta$. More specifically we will find the duality $L^\infty(U) - L^1(U)$ on a small enough neighborhood U of zero, with additional continuity at zero. First notice that equivalent to (strict5), is to show that $g = (\varphi u) * \check{v} - c$ is bounded around zero and continuous at zero. To facilitate L^1 test functions, define

$$U_\varepsilon = \left\{\psi \in \mathcal{D}(\mathbb{R}^n) : \|\psi\|_{L^1(\mathbb{R}^n)} \leq 1, \operatorname{supp}(\psi) \subset B(0, \varepsilon)\right\}.$$

The bounds we want to prove are

$$\forall \mu > 0, \exists \varepsilon > 0, \forall \psi \in U_\varepsilon : |\langle g, \psi\rangle| \leq \mu. \tag{2.13}$$

Suppose for a contradiction that (2.13) does not hold. Then we find $\mu > 0$ and a sequence $\{\varepsilon_j\}_{j \in \mathbb{N}} \to 0$ with $\psi_j \in U_{\varepsilon_j}$ such that $|\langle g, \psi_j\rangle| > \mu$ for all j. Sequential compactness of $[0, 1]$ and $0 \leq |\int \psi| \leq 1$ produces a subsequence $\{\tilde{\varepsilon}_j\}_{j \in \mathbb{N}}$ of $\{\varepsilon_j\}_{j \in \mathbb{N}}$ with $\alpha_j = |\int \tilde{\psi}_j| \to \alpha$ as $j \to \infty$ and $\tilde{\psi} \in U_{\tilde{\varepsilon}_j}$. For ease of notation, we drop the tilde. We have $\alpha \in \mathbb{C}$ and $|\alpha| \leq 1$. If $\alpha \neq 0$ then $\{\frac{\psi_j}{\alpha}\}_{j \in \mathbb{N}}$ is almost a strict delta sequence analagously to (2.11). By (2.12) it follows that $\langle g, \frac{\psi_j}{\alpha}\rangle \to 0$ as $j \to \infty$. This contradicts the negation of (2.13). If $\alpha = 0$, then consider

$$\lim_{j \to \infty} \langle g, \psi_j\rangle = \lim_{j \to \infty} \langle g, \psi_j + \sigma_j\rangle - \lim_{j \to \infty} \langle g, \sigma_j\rangle,$$

for any delta sequence $\{\sigma_j\}_{j \in \mathbb{N}}$. By (2.12) both terms are zero wich contradicts (2.13).

Using (2.13) for $\mu = 1$, we find some $\eta > 0$ such that

$$|\langle g, \psi\rangle| \leq 1, \quad \text{for all } \psi \in U_\eta.$$

This means that $g|_{B_\eta}$ is a functional on $\mathcal{D}(B_\eta)$, when topologised with the L^1-norm. Thus neccesarily $g \in L^\infty(B_\eta)$. This is the first property of (strict5). For continuity at zero we use (2.13) for succesively smaller μ.

$$\|g\|_{L^\infty(B_\varepsilon)} = \sup_{\psi \in U_\varepsilon} |\langle g, \psi\rangle| \to 0.$$

This implies that g is almost everywhere equal to a function continuous at zero. □

Next we give a characterization for strict product (strict4).

Theorem 2.6.3. *Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$. The following are equivalent*

(i) The strict product $[u \cdot v] \in \mathcal{D}'(\mathbb{R}^n)$ exists.

(ii) The limit $\lim_{\varepsilon \to 0} u(v * \rho^\varepsilon) + (u * \rho^\varepsilon)v$ exists in $\mathcal{D}'(\mathbb{R}^n)$ for every strict delta net $(\rho^\varepsilon)_{\varepsilon \in (0,1]}$.

(iii) For all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, the function $(\varphi u) * \check{v} + \check{u} * (\varphi v)$ is bounded in a neighborhood $\Omega \subseteq \mathbb{R}^n$ of 0 and is continuous at 0.

When these properties hold, we have

$$\begin{aligned}\langle [u \cdot v], \varphi\rangle &= \frac{1}{2} \lim_{\varepsilon \to 0} \langle u(\rho^\varepsilon * v) + (u * \rho^\varepsilon)v, \varphi\rangle \\ &= \frac{1}{2}\left((\varphi u) * \check{v} + \check{u} * (\varphi v)\right)(0)\end{aligned}$$

Proof. Assume that $[uv]$ exists. For $(\rho^\varepsilon)_{\varepsilon\in(0,1]}$ and $(\sigma^\varepsilon)_{\varepsilon\in(0,1]}$ strict delta nets, $(\frac{\rho^\varepsilon+\sigma^\varepsilon}{2})_{\varepsilon\in(0,1]}$ is a strict delta net. A simple calculation reveals an equivalent form for strict product (strict4).

$$
\begin{aligned}
[uv] &= \lim_{\varepsilon\to 0}(u*(\frac{\rho^\varepsilon+\sigma^\varepsilon}{2}))(v*(\frac{\rho^\varepsilon+\sigma^\varepsilon}{2})) && (2.14)\\
\iff 4[uv] &= \lim_{\varepsilon\to 0}(u*\rho^\varepsilon)(v*\rho^\varepsilon) && (2.15)\\
&\quad + (u*\rho^\varepsilon)(v*\sigma^\varepsilon)+(u*\sigma^\varepsilon)(v*\rho^\varepsilon)+(u*\sigma^\varepsilon)(v*\sigma^\varepsilon) && (2.16)\\
&= [uv] + \lim_{\varepsilon\to 0}(u*\rho^\varepsilon)(v*\sigma^\varepsilon)+(u*\sigma^\varepsilon)(v*\rho^\varepsilon)+[uv] && (2.17)\\
\iff [uv] &= \frac{1}{2}\lim_{\varepsilon\to 0}(u*\rho^\varepsilon)(v*\sigma^\varepsilon)+(u*\sigma^\varepsilon)(v*\rho^\varepsilon), && (2.18)
\end{aligned}
$$

for all strict delta nets $(\rho^\varepsilon)_{\varepsilon\in(0,1]}$ and $(\sigma^\varepsilon)_{\varepsilon\in(0,1]}$. On the other hand, if the limit (2.18) exists, we can choose $\rho^\varepsilon = \sigma^\varepsilon$.

An argument fully analogous to Theorem 2.6.2 then reduces (2.18) to a double limit, which implies (ii), and the converse is (iii) is proven by applying the proof of (strict5) to both terms simultaniously. □

Example 2.6.4. The strict product

$$\text{v.p.}\frac{1}{x}[\delta]$$

does not exist. Applying Theorem 2.6.2, we need

$$\varphi(x)\text{v.p.}\frac{1}{x}*\check{\delta} = \varphi(x)\text{v.p.}\frac{1}{x}$$

to be continuous at zero and bounded around zero, but it is neither. When both factors are regularised, we get the strict product

$$[\text{v.p.}\frac{1}{x}\cdot\delta] = \frac{1}{2}\delta'(x). \tag{2.19}$$

We employ the result of Theorem 2.6.3. For $\varphi\in\mathcal{D}(\mathbb{R})$,

$$
\begin{aligned}
\frac{1}{2}\left(\varphi(x)\text{v.p.}(\frac{1}{x})*\check{\delta}(x)+\varphi(x)\delta(x)*(\text{v.p.}\frac{1}{x})\check{}\right) &= \frac{1}{2}\left(\varphi(x)\text{v.p.}\frac{1}{x}+\varphi(0)\text{v.p.}\frac{-1}{x}\right)\\
&= \frac{1}{2}\left((\varphi(x)-\varphi(0)\right)\text{v.p.}\frac{1}{x}.
\end{aligned}
$$

It remains to notice that

$$(\varphi(x)-\varphi(0))\text{v.p.}\frac{1}{x} = \frac{\varphi(x)-\varphi(0)}{x},$$

is continuous at 0 and bounded around 0. We thus have

$$\langle[\text{v.p.}\frac{1}{x}\cdot\delta],\varphi\rangle = \lim_{x\to 0}\frac{\varphi(x)-\varphi(0)}{2x} = \lim_{x\to 0}\frac{\varphi'(x)}{2} = \frac{\varphi'(x)}{2} = \langle\frac{1}{2}\delta'(x)\rangle,$$

which is (2.19). △

The regularisation method used by the strict product can be used for other classes of delta nets. The set of delta nets considered is then implies the properties of the resulting product. One important such product is the model product.

2.7 Model product

The model product uses the class of model delta nets.

Definition 2.7.1 (Model delta net)**.** The class of model delta nets consists of the nets $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$, where

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right),$$

for any $\varphi\in\mathcal{D}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\varphi(x)dx = 1$.

Every model delta net is a strict delta net since have the support property

$$\operatorname{supp}\varphi_\varepsilon = \varepsilon \operatorname{supp}\varphi,$$

and the integral

$$\int_{\mathbb{R}^n} \varphi_\varepsilon(x)dx = \int_{\mathbb{R}^n} \varphi(x)dx,$$

and similarly for the integral of the absolute values. To create a model product, we regularise the multiplication of distributions by convolution with model delta nets as follows.

Definition 2.7.2 (Model product). For $u, v \in \mathcal{D}'(\mathbb{R}^n)$, we write

$$u[v] = \lim_{\varepsilon\to 0} u(\varphi_\varepsilon * v), \tag{model1}$$

$$[u]v = \lim_{\varepsilon\to 0} (\varphi_\varepsilon * u)v, \tag{model2}$$

$$[u][v] = \lim_{\varepsilon\to 0} (\varphi_\varepsilon * u)(\psi_\varepsilon * v), \tag{model3}$$

$$[uv] = \lim_{\varepsilon\to 0} (\varphi_\varepsilon * u)(\varphi_\varepsilon * v), \tag{model4}$$

if the limits exist and are equal for all model delta nets $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$.

The model product has an equivalent formulation similar to the condition (strict5). The difference is in the notion of value at a point.

Definition 2.7.3 (Value at a point, Lojasiewicz). A distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ has the value $c \in \mathbb{C}$ at the point $x_0 \in \mathbb{R}^n$ if the limit

$$\lim_{\varepsilon\to 0} \langle u(x), \varphi_\varepsilon(x - x_0)\rangle \tag{2.20}$$

exists and equals c for all model delta nets φ_ε.

Lojasiewicz introduced this concept in [26] as the value c such that for $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\lim_{\varepsilon\to 0} \langle u(x_0 + \varepsilon x), \varphi(x)\rangle = c \int_{\mathbb{R}^n} \varphi(x)dx.$$

It is clear that the value of (2.20) is equivalent to the value of the model product $u[\delta]$. This notion gains even more strength when combined with tensor products.

Definition 2.7.4. Let $u(x, y) \in \mathcal{D}'(\mathbb{R}^{2n})$ be a distribution of two variables, then u is said to have the section $T(x) \in \mathcal{D}'(\mathbb{R}^n)$ at $y = 0$ if $u(x, y)[1(x) \otimes \delta(y)] = T(x)$.

In [23] Jelínek proves a useful characterisation for model product (model4) by sections.

Theorem 2.7.5. *Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$. Model product* (model4) *exists if and only if*

$$\frac{1}{2}\left(u(x + y)v(x - y) + u(x - y)v(x + y)\right),$$

has a section $w(x)$ at $y = 0$ and then

$$w(x) = [u \cdot v].$$

2.8 Consistency of products of distributions

We now have defined several different product maps. We want to make sure that different methods gives the same products. As noted before, the model product extends the strict product. We prove that the strict product extends the Fourier product and extends the Duality product for spaces that are closed under convolution.

Proposition 2.8.1. *Let $V, W \subset \mathcal{D}'$ be subspaces of distributions such that the multiplication map $M : V \times W \to \mathcal{D}'$ is seperately continuous. Suppose that V is closed under convolution by elements of $\mathcal{D}$, then strict products* (strict1)-(strict3) *extend the multiplication map M.*

Proof. Let $v \in V$ and $w \in W$. The strict product is defined as the limit of $(\rho^\varepsilon * v)w$. We have that $\rho^\varepsilon * v \to v \in V$. By continuity of M, we conclude that $(\rho^\varepsilon * v)w$ converges to $M(v, w)$. □

For the duality prodcut on the Sobolev spaces, this theorem applies. This means that the strict product extends the Sobolev product maps given in Theorem 2.4.8. We now want to show the consistency of the strict product with the Fourier product. For this we need the following lemma.

Lemma 2.8.2. *Let $f \in \mathcal{D}'_{L^1}$, then $\mathcal{F}f$ is continuous.*

Proof. Represent f as $f = \sum_{|\alpha|\leq m} \partial^\alpha f_\alpha$ for $m \in \mathbb{N}$ and $f_\alpha \in L^1$. We compute

$$\mathcal{F}f = \sum_{|\alpha|\leq m} x^\alpha \mathcal{F}f_\alpha,$$

where every $\mathcal{F}f_\alpha$ is continuous by the Riemann-Lesbesgue theorem, Theorem 1.3.21, as the Fourier transform of an L^1 function. □

This allows us to prove that the strict product of the Fourier transforms coincides with the exchange formula (1.7) for when the S'-convolution exists.

Lemma 2.8.3. *Let $S, T \in \mathcal{S}'$. Assume that the $\mathcal{S}'$-convolution of S and T exists. Then strict product* (strict1)-(strict3) *of $\mathcal{F}S$ and $\mathcal{F}T$ exists and this strict product equals*

$$\mathcal{F}S[\mathcal{F}T] = \mathcal{F}(S * T). \tag{2.21}$$

Proof. By definition of the $\mathcal{S}'$-convolution the lefthandside of (2.21) is to be interpreted as

$$\begin{aligned} {}_{\mathcal{S}'}\langle \mathcal{F}(S * T), \varphi\rangle_{\mathcal{S}} &= {}_{\mathcal{S}'}\langle S * T, \mathcal{F}\varphi\rangle_{\mathcal{S}} && (2.22)\\ &= {}_{\mathcal{D}'_{L^1}}\langle (\mathcal{F}(\varphi) * \check{S})T, 1\rangle_{\mathcal{D}_{L^\infty}} && (2.23)\\ &= \mathcal{F}^{-1}((\mathcal{F}(\varphi) * \check{S})T)(0). && (2.24)\end{aligned}$$

In the last line we only have equality of values in case the $\mathcal{S}'$-convolution exists. Moreover, we have that $(\mathcal{F}(\varphi) * \check{S}) \in \mathcal{O}_M$ since the convolution is of type $\mathcal{S} * \mathcal{S}'$. Therefore we can apply the exchange formula to get

$$\mathcal{F}^{-1}((\mathcal{F}\varphi * \check{S})T) = \mathcal{F}^{-1}(\mathcal{F}\varphi * \check{S}) * \mathcal{F}^{-1}T = (\varphi\mathcal{F}S) * (\mathcal{F}T)\check{}.$$

By lemma 2.8.2, this function is continuous since $(\mathcal{F}\varphi * \check{S})T \in \mathcal{D}'_{L^1}$. Particularly it satisfies property (strict5), thus by Theorem 2.6.2 we get existence of the strict product $\mathcal{F}S[\mathcal{F}T]$ and equality to the value $\mathcal{F}^{-1}((\mathcal{F}(\varphi) * \check{S})T)(0)$.

□

Theorem 2.8.4. *Let $u, v \in \mathcal{D}'(\Omega)$ such that their Fourier product exists. Then the strict product (strict1)-(strict3) exists and equals the Fourier product.*

Proof. By definition of the Fourier product, we have existence of the $\mathcal{S}'$ convolution of $\mathcal{F}(u)$ and $\mathcal{F}(v)$. The Fourier product is then defined on Ω as $\mathcal{F}^{-1}(\mathcal{F}u * \mathcal{F}v)$. By Lemma 2.8.3, we have that

$$u[v] = \mathcal{F}^{-1}\mathcal{F}u[\mathcal{F}^{-1}\mathcal{F}v] = \mathcal{F}^{-1}(\mathcal{F}u * \mathcal{F}v).$$

□

We show off a neat observation related to Lemmas 2.8.2 and 2.8.3.

Corollary 2.8.5. *Let $u \in \mathcal{D}'$, then the Fourier product δu exists then u is continuous in a neighborhood of zero.*

Proof. We only need to consider the localized product in a neighborhood of zero as $\operatorname{supp}\delta = 0$. So let Ω be a neighborhood of zero and let $\varphi \in \mathcal{D}$ equal one on Ω. By existence of the Fourier product, there is some Ω small enough such that the $\mathcal{S}'$-convolution of $\mathcal{F}(\varphi\delta)$ and $\mathcal{F}(\varphi u)$ exists. We calculate that $\mathcal{F}(\varphi\delta) = \mathcal{F}(\varphi(0)\delta) = \mathcal{F}(\delta) = 1$. By definition of the $\mathcal{S}'$-convolution we have for all $\psi \in \mathcal{S}$

$$(\psi * \check{1})\mathcal{F}(\varphi u) = \left(\int \psi\right) \mathcal{F}(\varphi u) \in \mathcal{D}'_{L^1}.$$

Thus $\mathcal{F}(\varphi u) \in \mathcal{D}'_{L^1}$. We can then write $\mathcal{F}(\varphi u) = \sum_{|\alpha|\leq m} \partial^\alpha u_\alpha$ for some $m \in \mathbb{N}$ and $u_\alpha \in L^1$. Applying lemma 2.8.2 for the inverse Fourier transform makes φu continuous. We conclude that u is continuous in a neighborhood of zero. □

So far we have thus seen that several products can be reduced to the study of point values of distributions. This completes our discission of intrinsic products. We continue the chapter with a discussion of extrinsic multiplication, the Colombeau algebra and the very weak solution concept.

2.9 Extrinsic multiplication and Colombeau algebra

Another approach is to define an extrinsic multiplication. We want to have an associative algebra in which every distribution is represented. We then could perform the multiplication inside the algebra. If we are lucky the result might again represent a distribution. Ideally we want an algebra $(\mathcal{A}, +, \circ)$ with the following properties:

(i) $\mathcal{A}$ is associative.

(ii) $\mathcal{D}'$ is linearly embedded into $\mathcal{A}$ and the constant function 1 is the unity in $\mathcal{A}$.

(iii) There are derivation operators ∂_{x_i} which satisfie the Leibniz rule.

(iv) The operators ∂_{x_i} coincide with the usual partial derivatives.

(v) $\circ$ coincides with the pointwise product for continuous functions.

However, it is impossible to get all of these properties. This famous impossibility result is due to Schwartz [33].

Theorem 2.9.1. *If the algebra* $(\mathcal{A}, +, \circ)$ *satisfies* $(i), (ii), (iii), (iv)$*, then* (v) *does not hold.*

Proof. We give a contradicting example in one dimension. For multiple dimensions, consider the tensor product with the constant function 1. First define $x_+ = xH(x)$. Then $(i) - (iv)$ imply either

$$x_+ \circ x \neq x_+ \circ x_+, \quad \text{or} \tag{2.25}$$

$$x \circ (x \log|x| - x) \neq x^2 \log|x| - x^2. \tag{2.26}$$

For a contradiction, assume that both (2.25) and (2.26) are false. Using only properties (i)-(iv) the following calculations are valid.

$$\begin{aligned}
\partial^2(x_+) \circ x &= \partial^2(x_+ \circ x) - 2\partial(x_+)\partial(x) - x_+ \circ \partial^2(x) \\
&= \partial(x_+^2) - 2\partial x_+ = 0, \\
x \circ \partial^2(x \log|x| - x) &= \partial^2(x^2 \log|x| - x^2) - 2\partial(x \log|x| - x) \\
&= \partial(2x \log|x| - x) - \partial(2x \log|x| - 2x) = 1.
\end{aligned}$$

By associativity,

$$\partial^2(x_+) \circ 1 = \partial^2(x_+) \circ x \circ \partial^2(x \log|x| - x) = 0,$$

which contradicts (ii) because $\partial^2(x_+) = \delta(x)$ in $\mathcal{D}'$. □

The proof above shows that the problem is not with distributions but with pointwise multiplication of continuous functions. It is possible to define an associative algebra that satisfies (i)-(iv) and

(v') $\circ$ coincides with the product on smooth functions.

Colombeau algebra

We shall give a short introduction to the special Colombeau algebra $\mathcal{G}^s(\Omega)$ based on [14] and [20]. Colombeau algebras are a large topic with broad literature and numerous applications to PDE's. For the interested reader, we refer to the orginial works of Colombeau [10] and the work of Grosser et al. [20].

Let $\Omega \subset \mathbb{R}^n$ open. The space $C^\infty(\Omega)^{(0,1]}$ is the differential algebra of all maps from the interval $(0, 1]$ int $C^\infty(\Omega)$. The subalgebra of moderate families $\mathcal{E}_M(\Omega)$ consists of all nets $(u_\varepsilon)_{\varepsilon \in (0,1]} \in C^\infty(\Omega)^{(0,1]}$ such that for all $j \in \mathbb{N}$ and $K \subseteq \Omega$ compact, there are $C, N \geq 0$ with

$$|u_\varepsilon|_{j,K} \leq C\varepsilon^{-N},$$

where $|\cdot|_{j,K}$ are the $C^\infty(\Omega)$ seminorms (1.5). Every distribution $u \in \mathcal{D}'(\Omega)$ has a regularisation $(u_\varepsilon)_{\varepsilon \in (0,1]}$ in $\mathcal{E}_M(\Omega)$ trough convolution with a model delta net. We discuss this more in section 2.10. The ideal of negligible families $\mathcal{N}^s(\Omega)$ consists of the nets $(u_\varepsilon)_{\varepsilon\in(0,1]} \in C^\infty(\Omega)^{(0,1]}$ such that for all $j \in \mathbb{N}$ and $K \subset \Omega$ compact and all $N \geq 0$ there is $C \geq 0$ such that

$$|u_\varepsilon|_{j,K} \leq C\varepsilon^N.$$

The special Colombeau algebra then is

$$\mathcal{G}^s(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}^s(\Omega),$$

An element of the special Colombeau algebra $\mathcal{G}^s(\Omega)$ is then an equivalence class of moderate families. A moderate family $(u_\varepsilon)_{\varepsilon\in(0,1]} \in \mathcal{E}_M(\Omega)$ is called a representative of the Colombeau generalised function $U = (u_\varepsilon)_{\varepsilon\in(0,1]} + \mathcal{N}^s(\Omega)$.

Definition 2.9.2. A generalised function $U \in \mathcal{G}^s(\Omega)$ is associated with a distribution $u \in \mathcal{D}'(\Omega)$ if it has a representative $(u_\varepsilon)_{\varepsilon\in(0,1]} \in \mathcal{E}_M(\Omega)$ such that $u_\varepsilon \to u \in \mathcal{D}'(\Omega)$.

We show how to construct a Colombeau generalised function associated with a distribution. We adapt Proposition 1.2.20 of [20] where an explicit embedding of $\mathcal{E}'(\Omega)$ into $\mathcal{G}^s(\Omega)$ is shown.

Theorem 2.9.3. *Let $\Omega \subseteq \mathbb{R}^n$ and $u \in \mathcal{E}'(\Omega)$. Let $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ be a model delta net, then $u_\varepsilon = (u * \varphi_\varepsilon)|_\Omega$ satisfies*

$$\forall K \subseteq \Omega \text{ compact}, \forall \alpha \in \mathbb{N}^n,\ \exists M_\alpha, N_\alpha > 0,\ \sup_{x\in K} |\partial_x^\alpha u_\varepsilon| \leq M_\alpha \varepsilon^{-N_\alpha}.$$

Proof. By the structure theorem of compactly supported distributions, Theorem 1.3.12, there exists $r \in \mathbb{N}$, $c_\alpha \in \mathbb{C}$ and $f \in C(\Omega)$ continuous on Ω and compactly supported such that $u = \sum_{|\alpha|\leq r} c_\alpha \partial^\alpha f$. We show that each of the terms $v_\alpha = \partial^\alpha f$ satisfy the bound

$$\sup_{x\in K} |v_{\alpha,\varepsilon}| \leq C_\alpha \varepsilon^{-L_\alpha},$$

for some $C_\alpha, L_\alpha \geq 0$. It then follows that

$$\sup_{x\in K} |u_\varepsilon| \leq \sum_{|\alpha|\leq r} c_\alpha C_\alpha \varepsilon^{-L_\alpha} \leq M_0 \varepsilon^{-N_0},$$

with the constants

$$M_0 = \sum_{|\alpha|\leq r} c_\alpha C_\alpha,$$
$$N_0 = \max_{|\alpha|\leq r} L^\alpha.$$

So let $v_\alpha = \partial^\alpha f$. Fix $K \subset \Omega$ compact and take $\varepsilon \leq \varepsilon_0$ small enough. We have for any $x \in K$

$$\begin{aligned}
(v * \varphi_\varepsilon)(x) &= (f * \partial^\alpha \varphi_\varepsilon)(x) \\
&= \int f(x-y)\partial_x^\alpha \varphi_\varepsilon(y)dy \\
&= \int f(x-y)\varepsilon^{-n-|\alpha|}(\partial^\alpha \varphi)(\frac{y}{\varepsilon})dy \\
&= \int f(x-\varepsilon y)\varepsilon^{-|\alpha|}\partial_y^\alpha \varphi(y)dy \\
&\leq C_\alpha \varepsilon^{-|\alpha|},
\end{aligned}$$

Rn with $C_\alpha = ||f||_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} \left|\partial^\alpha \varphi(y)\right| dy$. The same argument applies for the estimate of the derivatives $\partial^\beta u$ since

$$\partial^\beta(u * \partial^\alpha \varphi_\varepsilon) = u * \partial^{\alpha+\beta}\varphi_\varepsilon.$$

□

We can map every distribution $u \in \mathcal{E}'(\Omega)$ to a moderate family through regularisation with a model delta net.

$$\mathcal{E}'(\Omega) \to \mathcal{E}_M(\Omega) : u \mapsto (u * \varphi_\varepsilon)_{\varepsilon\in(0,1]}|_\Omega.$$

This map can be extended for $u \in \mathcal{D}'(\Omega$ through localisation, see [20, Theorem 1.2.13, 1.2.14]. This construction was long known as sequential approach of Mikusinski [27].

Example 2.9.4. Not every moderate family is a regularisation of a distribution. Examples include oscillating nets and nets of large growth.

a) The net of constant functions $(u_\varepsilon)_{\varepsilon\in(0,1]} = \{\sin(\frac{1}{\varepsilon})\}$ is a moderate family since it it uniformly bounded in epsilon. However it does not converge distributionally.

b) Let $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ be a model delta net. The net of squares $(\varphi_\varepsilon^2)_{\varepsilon\in(0,1]}$ is a moderate family. However in Example 2.1.1, we showed that it can't converge to a distribution. △

What is the benefit of constructing the factor algebra $\mathcal{G}^s(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}^s(\Omega)$ over just considering the moderate families $\mathcal{E}_M(\Omega)$? Consider delta nets of the type $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\frac{x}{\varepsilon})$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with the additional property that

$$\int_{\mathbb{R}^n} x^\alpha \varphi_\varepsilon(x)dx = 0, \quad \forall \alpha \in \mathbb{N}^n.$$

Then the map

$$C^\infty(\Omega) \to \mathcal{G}^s(\Omega) : f \mapsto f * \varphi_\varepsilon|_\Omega + \mathcal{N}^s(\Omega),$$

is independent of the delta net φ_ε chosen. This unique equivalence class also contains the constant embedding $f + \mathcal{N}^s(\Omega)$. It follows that the multiplication of the embeddings coincide with the pointwise multiplication of smooth functions.

Jelinek [23], proven for the full Colombeau algebra $\mathcal{G}(\Omega)$.

Theorem 2.9.5. *Let $U, V \in \mathcal{G}(\Omega)$ with associated distributions $u, v \in \mathcal{D}'(\Omega)$. Let $W = UV \in \mathcal{G}(\Omega)$ be the product in the special Colombeau algebra of U and V. Then W is associated to a distribution $w \in \mathcal{D}'(\Omega)$ if and only if the model product $[uv]$ exists and then $w = [uv]$.*

The theory of Colombeau algebras is can be used to analyse partial differential equations with singular coefficients and initial or boundary conditions. Examples from literature are [13], [14] and [21]. It is somehow still difficult to use it in practical applications since the initial data must be represented in the Colombeau algebra, which is often not the case. Recently the very weak solution has given another approach to solving PDEs with singular coefficients. Although the concepts of moderate and negligigle nets seem similar to the Colombeau setting, the very weak solution concept has the advantage for being applied directly to the PDE. Therefore its results can give a more refined treatment of the singular terms.

2.10 Very weak solution

The very weak solution concept orginates from the a 2015 article by Garetto and Ruzhansky [19]. Here the authors use the method of the very weak solution to analyse hyperbolic equations. The idea has since been applied more widely, see e.g. [2, 3, 4, 5, 28, 31, 32]. The method consists of three main parts: existence, uniqueness and consistency.

The very weak solution concept describes its solutions through the qualitative behaviour of nets of regularised solutions. Moderate and negligible nets are defined in the following way.

Definition 2.10.1 (Moderate and negligible nets)**.** Let $(a_\varepsilon)_{\varepsilon\in(0,1]}$ be a net of non-negative real numbers.

a) The net $(a_\varepsilon)_{\varepsilon\in(0,1]}$ is moderate if there are $C, N \geq 0$ such that

$$a_\varepsilon \leq C\varepsilon^{-N}.$$

b) The net $(a_\varepsilon)_{\varepsilon\in(0,1]}$ is negligible if for all $q \geq 0$ there are constants $C_q \geq 0$ such that

$$a_\varepsilon \leq C_q\varepsilon^q.$$

Remark 2.10.2. Suppose that the net $(a_\varepsilon)_{\varepsilon\in(0,1]}$ is moderate and that $(b_\varepsilon)_{\varepsilon\in(0,1]}$ is negligible, then

$$a_\varepsilon b_\varepsilon \leq C\varepsilon^{-N}C_q\varepsilon^q, \quad \text{for all } q > 0,$$

such that the net of products $(a_\varepsilon b_\varepsilon)_{\varepsilon\in(0,1]}$ is negligible.

The quantities that we use to describe the net behaviour are often norms or semi-norms of the function nets. It will be useful to have language for that.

Definition 2.10.3 (X-moderate, X-negligible nets)**.** Let X be a Banach space with norm $||\cdot||_X$.

a) A net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is X-moderate if the net of X-norms $(||u||_X)_{\varepsilon\in(0,1]}$ is moderate. A net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is X-negligible if the net of X-norms is negligible.

Let X be a Fréchet space with seminorms $\{|\cdot|_k, k\in\mathbb{N}\}$.

a) A net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is X-moderate if for all $k\in\mathbb{N}$ the nets of k-th seminorms $(|u_\varepsilon|_k)_{\varepsilon\in(0,1]}$ are moderate.

b) A net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is X-negligible if for all $k\in\mathbb{N}$ the nets of k-th seminorms $(|u_\varepsilon|_k)_{\varepsilon\in(0,1]}$ are negligible.

We also need nets of slower growth.

Definition 2.10.4 (Moderate of log-type)**.** Let $(a_\varepsilon)_{\varepsilon\in(0,1]}$ be a net of non-negative real numbers. The net $(a_\varepsilon)_{\varepsilon\in(0,1]}$ is moderate of log-type if there exists $C\geq 0$ such that

$$a_\varepsilon \leq C\log\frac{1}{\varepsilon}.$$

Let X be a normed space. A net $(u_\varepsilon)_{\varepsilon\in(0,1]}\subset X$ is X-moderate of log-type if the net of norms $(||u_\varepsilon||_X)_{\varepsilon\in(0,1]}$ is moderate of log-type. Similarly for a Fréchet space if for all seminorms the net of seminorms is moderate of log-type.

Example 2.10.5. We show how to construct log-type moderate nets from moderate nets. Suppose that $(a_\varepsilon)_{\varepsilon\in(0,1]}$ is a moderate net. We change the parametrisation of ε to get a slower rate of growth. By moderateness of $(a_\varepsilon)_{\varepsilon\in(0,1]}$ we have the estimate

$$a_\varepsilon \leq C_1\varepsilon^{-N}, \tag{2.27}$$

for constants $C_1, N\geq 0$. Now consider the net.

$$(b_\varepsilon)_{\varepsilon\in(0,1]} = (a_{\lambda_\varepsilon})_{\varepsilon\in(0,1]},$$

where

$$\lambda_\varepsilon = \left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{N}}. \tag{2.28}$$

We show that $(b_\varepsilon)_{\varepsilon\in(0,1]}$ is moderate of log-type. By (2.27) we can estimate for b_ε

$$\begin{aligned} b_\varepsilon = a_{\lambda_\varepsilon} &\leq C_1\lambda_\varepsilon^{-N}, \\ &= C_1\left(\left(\log\frac{1}{\varepsilon}\right)^{-\frac{1}{N}}\right)^{-N} = C_1\log\frac{1}{\varepsilon}, \end{aligned}$$

which proves that $(b_\varepsilon)_{\varepsilon\in(0,1]}$ is moderate of log type.

If the constant N is unknown, then one can use the following reparametrisation. Let

$$c_\varepsilon = (a_{\mu_\varepsilon})_{\varepsilon\in(0,1]},$$

where

$$\mu_\varepsilon = \frac{1}{\log\log\frac{1}{\varepsilon}}.$$

We notice that fact that

$$(\log x)^N \leq x,$$

for $x\in\mathbb{R}$ sufficiently large. Then $(c_\varepsilon)_{\varepsilon\in(0,1]}$ is moderate of log-type since for any $N\geq 0$ it holds that

$$\begin{aligned} c_\varepsilon &= a_{\mu_\varepsilon} \\ &\leq C_1\left(\frac{1}{\log\log\frac{1}{\varepsilon}}\right)^{-N}, \\ &\leq C_1\left(\log\log\frac{1}{\varepsilon}\right)^{N}, \\ &\leq C_1\log\frac{1}{\varepsilon}, \end{aligned}$$

if ε is sufficiently small. $\triangle$

Example of a very weak solution

Now we illustrate the very weak solution concept through a simple example. We consider the boundary value problem (1.15) but with distributional righthandside $f \in \mathcal{D}'((0,1))$,

$$\begin{cases} \frac{d}{dx}u(x) - \frac{d^2}{dx^2}u(x) = f(x) \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{2.29}$$

If we take some regularisation $(f_\varepsilon)_{\varepsilon\in(0,1]}$ of f, then we can consider for each $\varepsilon \in (0,1]$ the regularised problem

$$\begin{cases} \frac{d}{dx}u_\varepsilon(x) - \frac{d^2}{dx^2}u_\varepsilon(x) = f_\varepsilon(x), \\ u_\varepsilon(0) = 0, \quad u_\varepsilon(1) = 0. \end{cases} \tag{2.30}$$

For $f_\varepsilon \in L^2((0,1))$ the regularised problem has the weak formulation:

$$\int_0^1 \frac{d}{dx}u_\varepsilon(x)v(x)dx + \int_0^1 \frac{d}{dx}u_\varepsilon(x)\frac{d}{dx}v(x)dx = \int_0^1 f_\varepsilon(x)v(x)dx, \quad \forall v \in H^1((0,1)), \tag{2.31}$$

with the solution $u \in H_0^1((0,1))$.

Definition 2.10.6. We call a net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ a very weak solution to (2.29) if:

- The net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is in $H_0^1((0,1))$.
- There is a regularisation $(f_\varepsilon)_{\varepsilon\in(0,1]}$ of f such that for each $\varepsilon \in (0,1]$ the function u_ε solves the weak regularised problem (2.31).
- The net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2((0,1))$-moderate.

In this definition the moderateness of the solution net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is subject to change. For example one can consider a stronger notion where the net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ admits a stronger type of moderateness. Depending on the form of the equation and the energy estimate, it may be neccesary to ask stronger moderateness for some of the coefficients.

We prove the existence of a very weak solution for problem (2.29). Let $(\varphi_\varepsilon)_{\varepsilon\in(0,1]}$ be a model delta net. Then $(f_\varepsilon)_{\varepsilon\in(0,1]} = (f * \varphi_\varepsilon|_{(0,1)})_{\varepsilon\in(0,1]}$ is an $L^2((0,1))$-moderate regularisation of f. Without proof, by the Lax-Milgram theorem we find for each $\varepsilon \in (0,1]$ a unique solution $u_\varepsilon \in H_0^1((0,1))$ to the weak regularised problem (2.31). If the net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2((0,1))$-moderate, then it is a very weak solution to (2.29). Also by the Lax-Milgram theorem, we have the energy estimate

$$||u_\varepsilon||_{L^2((0,1))} \leq \frac{1}{c}||f_\varepsilon||_{L^2((0,1))},$$

where c is the ellipticity constant of the bilinear form. Since $(f_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2((0,1))$-moderate, then so is $(u_\varepsilon)_{\varepsilon\in(0,1]}$.

Let's now discuss uniqueness of the very weak solution.

Definition 2.10.7. We say that the problem (2.29) has a very weak solution if the following holds. Suppose $(u_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ are two very weak solutions to (2.29) by respective regularisations $(g_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\tilde{g}_\varepsilon)_{\varepsilon\in(0,1]}$ of g. If

$$(g_\varepsilon - \tilde{g}_\varepsilon)_{\varepsilon\in(0,1]} \text{ is } L^2((0,1))\text{-negligible},$$

then

$$(u_\varepsilon - \tilde{u}_\varepsilon)_{\varepsilon\in(0,1]} \text{ is } L^2((0,1))\text{-negligible}.$$

In the case of the example problem (2.29) we use again the energy estimate. We have that the very weak solution $(u_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies equation (2.31) for each ε and that $(\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ satisfies (2.31) with righthandside $\tilde{f}_\varepsilon$. Substracting equations gives

$$\int_0^1 \frac{d}{dx}\left(u_\varepsilon(x) - \tilde{u}_\varepsilon(x)\right) v(x)dx + \int_0^1 \frac{d}{dx}\left(u_\varepsilon(x) - \tilde{u}_\varepsilon(x)\right) \frac{d}{dx} v(x)dx = \int_0^1 \left(f_\varepsilon(x) - \tilde{f}_\varepsilon(x)\right) v(x)dx,$$

for all $v \in H^1((0,1))$. The Lax-Milgram theorem is again applicable, now with the righthandside $f_\varepsilon - \tilde{f}_\varepsilon$ and solution $u_\varepsilon - \tilde{u}_\varepsilon$. Therefore we have the energy estimate

$$||u_\varepsilon - \tilde{u}_\varepsilon||_{L^2((0,1))} \leq \frac{1}{c}||g_\varepsilon - \tilde{g}_\varepsilon||_{L^2((0,1))}.$$

Given that $g_\varepsilon - \tilde{g}_\varepsilon$ is $L^2((0,1))$-negligible, we conclude that $u_\varepsilon - \tilde{u}_\varepsilon$ is $L^2((0,1))$-negligible.

In the next chapter we will apply the very weak solution concept to a concrete problem.

Chapter 3

The Euler-Bernoulli equation

In this chapter we define a very weak solution for the clamped Euler-Bernoulli equation with discontinuous cross-section and distributional forces and show its unique existence. In [21] the authors solve this equation in a Colombeau algebra. The goal is to adapt the results of [21] to the very weak solution setting. First we discuss the physical modelling of the Euler-Bernoulli beam. Next we present the weak formulation of the Euler-Bernoulli equation, following [21]. Then we prove the existence of an L^2-moderate very weak solution with minimal conditions on the regularisations of the coefficients. For the uniqueness of the very weak solution we need to assume additional moderateness of derivatives of the solution. In the last part of this chapter we numerically investigate the solutions to the Euler-Bernoulli equation with distributional forces.

3.1 Physical model

We discuss how the Euler-Bernoulli beam equation is derived as a model for an elastic rod. A general theory of elasticity is found in [7].

Let us consider a cylindrical beam of length L. A beam is often modelled by considering deviations from the beam axis. The beam is under influence of a vertical force g_1 and an axial force P. We assume that there are no lateral forces. We model the vertical displacement of the beam axis at time t and position x by the function $u(x,t)$.

Material particles also experience displacements in the axial direction $w(x,z,t)$. That is, suppose p is a particle that starts at the coordinates (x,z) at time $t=0$. Then $w(x,z,t)$ is the horizontal displacement of p at time t. For small horizontal deformations we can approximate $w(x,z,t) = z\partial_x u(x,t)$. We model the elasticity of the beam by Hooke's law. The stress $\sigma(x,t)$ is then proportional to the relative extension of the material. Locally, the relative extension is given by the derivative $\partial_x w(x,z,t)$. This is $\sigma(x,t) = E(x)\partial_x w(x,z,t)$. The proportionality constant $E(x)$ is called the modulus of elasticity. The bending moment M is a stress resultant given by $M(x,t) = \int_{C(x)} \sigma(x,t) z dA$, with the integral over the cross-section of the beam $C(x)$ at position x. In our case we find

$$
\begin{aligned}
M(x,t) &= \int_{C(x)} E(x)\partial_x w(x,z,t) z dA = \int_{C(x)} E(x)\partial_x^2 u(x,t) z^2 dA \\
&= E(x)\partial_x^2 u(x,t) \int_{C(x)} z^2 dy dz = E(x)I(x)\partial_x^2 u(x,t),
\end{aligned}
$$

where we put $I(x) = \int_{C(x)} z^2 dy dz$, which is the second moment of area of the beam's cross-section. A shearing force is defined as an excess force on the top part of the beam over the bottom part of the beam. The bending moment defines the shearing force

$$Q_1(x,t) = -\partial_x^2 M(x,t) = -\partial_x^2 \left(E(x)I(x)\partial_x^2 u(x,t) \right).$$

The minus sign is because an excess force in a position x with positive curvature results in a downwards force. Next we have the shearing force due to the axial force, that is

$$Q_2(x,t) = -P(t)\partial_x^2 u(x,t).$$

The shearing forces are vertical forces as a result of the stress and the axial force acting on a curved beam and thus deforming the beam axis. Using Newton's second law on the vertical displacement $u(x,t)$ and the vertical forces $g_1(x,t)$, $Q_1(x,t)$ and $Q_2(x,t)$ results in the dynamic Euler-Bernoulli equation

$$R(x)\partial_t^2 u = g_1(x,t) + Q_2(x,t) + Q_2(x,t). \tag{3.1}$$

The line density $R(x)$ is a replacement for the proper mass, as we are working locally on a cross-section of infinitesimal width. It is defined as $\partial_x m(x)$ with $m(x)$ the mass of the beam contained in the interval $[0,x]$. Rearranging equation (3.1) we get explicitly

$$\partial_x^2 \left(A(x)\partial_x^2 u\right) + P(t)\partial_x^2 u + R(x)\partial_t^2 u = g_1(x,t),$$

where

- A is the bending stiffness, given as $A(x) = E(x)I(x)$, with $E(x)$ the modulus of elasticity of the material and $I(x)$ the moment of inertia.
- R is the line density, i.e. mass density per unit length.
- P is the axial force.
- g_1 is the vertical force.
- u is the vertical displacement of the beam axis. Thus $\partial_t^2 u$ is the acceleration of the rod and ∂_x^2 is the linearized curvature.

We shall discuss solutions to the clamped Euler-Bernoulli beam, i.e. the boundary conditions

$$u(0,t) = u(1,t) = \partial_x u(0,t) = \partial_x u(1,t) = 0.$$

And we put the initial conditions

$$u(x,0) = f_1(x), \quad \partial_t u(x,0) = f_2(x).$$

We want to consider the case of a beam with discontinuous materials, such that A and R are of Heaviside type. For example $A(x) = EI_1 + H(x-x_0)(EI_2 - EI_1)$, for some $x_0 \in (0,1)$ that is the point of discontinuity. The modulus of elasticity of the material E is constant for the whole beam and I_1 and I_2 are the second moments of area of the left and right part respectively. For R we consider $R(x) = R_1 + H(x-x_0)(R_2 - R1)$, where R_1, R_2 are the line densities of the beam parts.

We want to discuss distributional forces $P(t) \in \mathcal{D}'([0,T])$ and $g_1(x,t) \in \mathcal{D}'([0,1]\times[0,T])$ to allow for forces like $P(t) = P_0 + P_1\delta(t-t_0)$ and $g_1(x,t) = F(t)\delta(x-x_1)$.

Standard theory of PDE's handle the case where the coeffient of ∂_t^2 is constant. Therefore we formally apply the change of variables $t \mapsto \sqrt{R(x)}t$ to (3.1) which gives

$$\partial_x^2 \left(A(x)\partial_x^2 u(x,\sqrt{R(x)}t)\right) + P(\sqrt{R(x)}t)\partial_x^2 u(x,\sqrt{R(x)}t) + R(x)\frac{\partial_t^2 u(x,\sqrt{R(x)}t)}{\sqrt{R(x)}^2} = g(x,\sqrt{R(x)}t).$$

This simplifies to the equation

$$\partial_x^2 \left(c(x)\partial_x^2 u\right) + b(x,t)\partial_x^2 u + \partial_t^2 u = g(x,t), \tag{3.2}$$

with the notation $c(x) = A(x), b(x,t) = P(\sqrt{R(x)}t), g(x,t) = g_1(x,\sqrt{R(x)}t)$ and the same name for u. We will further focus our analysis on the transformed equation (3.2) under the assumption that the change of variables $t \mapsto \sqrt{R(x)}t$ is well-defined. We start with a weak solution for L^∞ coefficients, which we will apply when solving regularisations of the Euler-Bernoulli equation with distributional coefficients.

3.2 Weak solution to the Euler-Bernoulli equation with L^∞ coefficients

The existence and uniqueness of a very weak solution to the Euler-Bernoulli problem has been given in [21] by Oparnica and Hörmann. They adapted abstract variational results for a time dependent weak formulation such that it can be applied to the case of the Euler-Bernoulli equation. A general theory for time-dependent weak formulations can be found in [12, chapter XVIII]. We specifically refer to chapter XVIII §5 p. 552 about evolution problems of second order in time. In what follows, we follow the approach of Sections 1.3 and 2 of [21]. First is a theorem for an abstract formulation. Then it is applied to the Euler-Bernoulli equation in Theorem 3.2.2.

Let V, H be two complex, seperable Hilbert [1] spaces, where V is densely embedded into H. Denote the norm in V by $|\cdot|$ and in H by $||\cdot||$. Thus if V^* is the anti-dual[2] of V, then $V \subset H \subset V^*$. Let $a(t,.,.)$, $a_0(t,.,.)$, and $a_1(t,.,.)$, $t \in [0,T]$, be families of continuous sequilinear[3] forms on V with

$$a(t,u,v) = a_0(t,u,v) + a_1(t,u,v), \quad \forall u,v \in V,$$

such that a_0 and a_1 satisfy

(i) for all $u, v \in V : t :\mapsto a_0(t,u,v)$ is continuously differentiable $[0,T] \to \mathbb{C}$,

(ii) a_0 is Hermitian, i.e. $a_0(t,u,v) = \overline{a_0(t,v,u)}$ for all $u,v \in V$,

(iii) there exist real constants λ and $\alpha > 0$ such that

$$a_0(t,u,u) \geq \alpha|u|^2 - \lambda||u||^2, \quad \forall u \in V, \forall t \in [0,T],$$

(iv) for all $u, v \in V : t \mapsto a_1(t,u,v)$ is continuous in $[0,T] \to \mathbb{C}$,

(v) there is $C_1 \geq 0$ such that for all $t \in [0,T]$ and $u,v \in V : |a_1(t,u,v)| \leq C_1|u|||v||$.

Theorem 3.2.1. *Let $a(t,.,.)$ satisfy conditions $(i)-(v)$. Let $u_0 \in V$, $u_1 \in H$, and $f \in L^2((0,T),H)$. Then there exists a unique solution $u \in L^2((0,T),V)$ satisfying the regularity conditions*

$$u' = \partial_t u \in L^2((0,T),V), \quad \textit{and} \quad u'' = \partial_t^2 \in L^2((0,T),V'),$$

and solving the abstract initial problem

$$\langle u''(t), v\rangle + a(t,u(t),v) = \langle f(t), v\rangle, \quad \forall v \in V, \forall t \in (0,T), \tag{3.3}$$
$$u(0) = u_0, \quad u'(0) = u_1. \tag{3.4}$$

Additionally, we have the energy estimate

$$|u(t)|^2 + ||u'(t)||^2 \leq \left(D_T|u_0|^2 + ||u_1|| + \int_0^t ||f(\tau)||^2 d\tau \right) \cdot \exp(tF_T), \quad \forall t \in [0,T],$$

where the constants D_T and F_T are given by

$$D_T = (C + \lambda(1+T))/\min(1,\alpha) \quad \textit{and} \quad F_T = \max(C_0 + C_1, C_1 + T + 2)/\min(1,\alpha).$$

We describe how this abstract formulation applies to the clamped Euler-Bernoulli beam equation. Let $H = L^2((0,1))$ and $V = H_0^2((0,1))$. We write the $L^2((0,1))$ inner product as $\langle u,v\rangle = \int_0^1 u(x)\overline{v(x)}dx$. The anti-dual of V is equal to its dual $V' = H^{-2}((0,1))$. We define the sequilinear form $a = a_0 + a_1$ on $V \times V$ by

$$a_0(t,u,v) = \langle \partial_x^2 u, \partial_x^2 v\rangle, \quad a_1(t,u,v) = \langle b(t)\partial_x^2 u, v\rangle, \quad u,v \in V.$$

It then remains to specify sufficient conditions on the coefficients b and c and the righthandside g such that Theorem 3.2.1 is applicable. From [21, Theorem 2.2], we have the weak solution.

Theorem 3.2.2. *Assume $b \in C([0,T], L^\infty((0,1))$ and $c \in L^\infty((0,1))$ with*

$$0 < c_0 \leq c(x) \leq c_1, \quad \textit{for } x \in (0,1),$$

[1] A Hilbert space is a normed space where the norm is induced by an inner product. A topological space is separable if it has a countable, dense subset.

[2] The anti-dual V^* is the space of anti-linear functionals f on V. The functional f is anti-linear if $f(\lambda v) = \bar{\lambda} f(v)$.

[3] A map of two arguments is sequilinear if it is linear in the first argument and anti-linear in the second

where $c_0, c_1 \in \mathbb{R}$ *are constants. Suppose* $f_1 \in H_0^2((0,1)), f_2 \in L^2((0,1))$ *and* $g \in L^2((0,T), L^2((0,1)))$, *then there exists a unique* $u \in L^2((0,T), H_0^2((0,1)))$ *that solves the initial value problem*

$$\langle \partial_t^2 u, v\rangle + \langle c\partial_x^2 u, \partial_x^2 v\rangle + \langle b(t)\partial_x^2 u, v\rangle = \langle g(t), v\rangle, \quad \forall v \in H_0^2((0,1)), t \in (0,T), \tag{3.5}$$

$$u(x,0) = f_1(x), \quad \partial_t u(x,0) = f_2(x). \tag{3.6}$$

Additionally we have

$$u \in H^1((0,T), H_0^2((0,1))) \cap H^2((0,T), H^{-2}((0,1))),$$

and for any $t \in [0,T]$ *the energy estimate*

$$||u(t)||^2_{H^2((0,1))} + ||\partial_t u(t)||^2_{L^2((0,1))} \leq \left(D_T ||f_1||_{H^2((0,1))} + ||f_2||_{L^2((0,1))} + \int_0^t ||g(\tau)||^2_{L^2((0,1))} d\tau \right) \exp(tF_T), \tag{3.7}$$

holds, where

$$D_T = (c_1 + C_{1/2} c_0 (1+T)) / \min(\frac{c_0}{2}, 1), \quad \textit{and} \quad F_T = (||b||_{L^\infty(X_T)} + T + 2) / \min(\frac{c_0}{2}, 1).$$

The constant $C_{1/2}$ *is such that*

$$||v||^2_{H^1((0,1))} \leq \frac{1}{2} ||v||^2_{H^2((0,1))} + C_{1/2} ||v||_{L^2((0,1))},$$

for all $v \in H_0^2((0,1))$.

Write the time-space domain $(0,T) \times (0,1)$ as X_T. In [21], the authors continue with a solution of the clamped Euler-Bernoulli equation the Colombeau algebra $\mathcal{G}_{H^\infty(X_T)}$. Essentially they consider $H^\infty(X_T)$ moderate nets for the coefficients and the initial conditions. In our very weak solution we consider distributional coefficients and fixed initial conditions.

3.3 Very weak solution to the Euler-Bernoulli equation

Given the initial values $f_1 \in H_0^2((0,1))$, $f_2 \in L^2((0,1))$ and the coefficients $b \in \mathcal{D}'(X_T)$, $c \in \mathcal{D}'((0,1))$ and $g \in \mathcal{D}'(X_T)$, then we consider the initial-boundary value problem

$$\partial_x^2 \left(c(x) \partial_x^2 u \right) + b(x,t) \partial_x^2 u + \partial_t^2 u = g(x,t), \tag{EB}$$

$$u(0,t) = u(1,t) = 0, \quad \partial_x u(0,t) = \partial_x u(1,t) = 0, \tag{bc}$$

$$u(x,0) = f_1(x), \quad \partial_t u(x,0) = f_2(x). \tag{ic}$$

We define what we mean by a very weak solution.

Definition 3.3.1 (Very weak solution)**.** A net $(u_\varepsilon)_{\varepsilon \in (0,1]}$ is a very weak solution to (EB), (bc), (ic) if

- there exist nets $(b_\varepsilon)_{\varepsilon \in (0,1]}$ regularisation of b, $(c_\varepsilon)_{\varepsilon \in (0,1]}$ regularisation of c and $(g_\varepsilon)_{\varepsilon \in (0,1]}$ regularisation of g such that,
- for each $\varepsilon \in (0,1]$ the function u_ε is a weak solution (3.5), (3.6), to the regularised problem

$$\partial_x^2 \left(c_\varepsilon(x) \partial_x^2 u_\varepsilon \right) + b_\varepsilon(x,t) \partial_x^2 u_\varepsilon + \partial_t^2 u_\varepsilon = g_\varepsilon(x,t), \tag{EB$_\varepsilon$}$$

$$u_\varepsilon(0,t) = u_\varepsilon(1,t) = 0, \quad \partial_x u_\varepsilon(0,t) = \partial_x u_\varepsilon(1,t) = 0, \tag{bc$_\varepsilon$}$$

$$u_\varepsilon(x,0) = f_1(x), \quad \partial_t u_\varepsilon(x,0) = f_2(x). \tag{ic$_\varepsilon$}$$

- The net $(u_\varepsilon)_{\varepsilon \in (0,1]}$ is $L^2(X_T)$-moderate.

The choice of regularisations for the coefficients b and c and righthandside c is still unspecified. For all distributions there exist C^∞-moderate regularisations through convolution with model delta nets, see Theorem 2.9.3. We want to consider more general classes of regularisations that are sufficient to guarantee that the solution net $(u_\varepsilon)_{\varepsilon \in (0,1]}$ is $L^2(X_T)$-moderate. We prove existence of a very weak solution first on the level of nets.

Theorem 3.3.2. *Let $f_1 \in H_0^2((0,1))$, $f_2 \in L^2((0,1))$. Let $(b_\varepsilon)_{\varepsilon\in(0,1]}$ be an $L^\infty(X_T)$-log type moderate net in $C([0,T], L^\infty((0,1))$. Let $(g_\varepsilon)_{\varepsilon\in(0,1]}$ be an $L^2((0,T), L^2((0,1))$-moderate net. and let $(c_\varepsilon)_{\varepsilon\in(0,1]} \subset L^\infty((0,1))$ be a net which satisfies the bounds*

$$0 < c_0 \le c_\varepsilon(x) \le c_1, \quad \textit{for all } x \in [0,1],$$

for positive constants c_0, c_1. Then for each $\varepsilon \in (0,1]$ there is a weak solution u_ε to the problem (EB$_\varepsilon$), (bc$_\varepsilon$), (ic$_\varepsilon$). *The net of solutions $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is L^2-moderate.*

Proof. For fixed $\varepsilon \in (0,1]$ a weak solution to the problem (EB$_\varepsilon$), (bc$_\varepsilon$), (ic$_\varepsilon$) is the weak formulation (3.5), (3.6). Additionally all conditions of Theorem 3.2.2 are verbatim fulfilled. For every $\varepsilon \in (0,1]$ we this find a unique

$$u_\varepsilon \in H^1((0,T), H_0^2((0,T))) \cap H^2((0,T), H^{-2}((0,1))),$$

which is a weak solution to the problem (EB$_\varepsilon$), (bc$_\varepsilon$), (ic$_\varepsilon$). We claim that the net of solutions $(u_\varepsilon)_{\varepsilon\in(0,1])}$ is $L^2(X_T)$-moderate. We use the energy estimate (3.7) of Theorem 3.2.2. For any $t \in [0,T]$ it holds

$$\begin{aligned} ||u_\varepsilon(t)||^2_{L^2((0,1))} &\le ||u(t)||^2_{H^2((0,1))} + ||\partial_t u(t)||^2_{L^2((0,1))}, \\ &\le \left(D_T^\varepsilon ||f_1||_{H^2((0,1))} + ||f_2||_{L^2((0,1))} + \int_0^t ||g(\tau)||^2_{L^2((0,1))} d\tau \right) \exp(tF_T^\varepsilon). \end{aligned}$$

Using the moderateness of $(b_\varepsilon)_{\varepsilon\in(0,1]}$ and $(c_\varepsilon)_{\varepsilon\in(0,1]}$, we find

$$D_T^\varepsilon \le C||c_\varepsilon||_{L^\infty((0,1))}) \le C', \quad \text{and } F_T^\varepsilon \le K||b||_{L^\infty((0,T)\times(0,1))}) \le K' \log(\frac{1}{\varepsilon}),$$

for constants $C, C', K, K' > 0$. Thus

$$\begin{aligned} ||u_\varepsilon(t)||^2_{L^2((0,1))} &\le \left(C' + \int_0^t ||g_\varepsilon(\tau)||^2_{L^2((0,1))} d\tau \right) \exp(tK' \log(\frac{1}{\varepsilon})), \\ &= \left(C' + \int_0^t ||g_\varepsilon(\tau)||^2_{L^2((0,1))} d\tau \right) \varepsilon^{-tK'}. \end{aligned}$$

This bound is monotonically increasing in t and thus reaches its maximum at $t = T$. Thus for all $t \in [0,T]$ it holds

$$\begin{aligned} ||u_\varepsilon(t)||^2_{L^2((0,1))} &\le \left(C' + \int_0^T ||g_\varepsilon(\tau)||^2_{L^2((0,1))} d\tau \right) \varepsilon^{-TK'} \\ &= \left(C' + ||g_\varepsilon||_{L^2((0,T),L^2((0,1))} \right) \varepsilon^{-TK'}. \end{aligned}$$

By $L^2((0,1), L^2((0,1))$-moderateness of $(g_\varepsilon)_{\varepsilon\in(0,1]}$ we have the bound

$$||g_\varepsilon||_{L^2((0,T),L^2((0,1))} \le C_g \varepsilon^{-N_g},$$

for some constants $C_g, N_g \ge 0$. So next we estimate

$$\begin{aligned} ||u_\varepsilon(t)||^2_{L^2((0,1))} &\le \left(C' + C_g \varepsilon^{-N_g} \right) \varepsilon^{-TK'}, \\ &\le C'' \varepsilon^{-K''}, \end{aligned} \tag{3.8}$$

for the constants $C'' = \max(C', C_g)$ and $K'' = \max(N_g, TK')$. Now integrating from 0 to T gives

$$\begin{aligned} ||u_\varepsilon||^2_{L^2((0,T),L^2((0,1)))} &= \int_0^T ||u_\varepsilon(\tau)||^2_{L^2((0,1))} d\tau, \\ &\le \int_0^T C'' \varepsilon^{-K''} d\tau, \\ &= TC'' \varepsilon^{-K''}. \end{aligned}$$

This proves $L^2((0,T),L^2((0,1)))$-moderateness of $(u_\varepsilon)_{\varepsilon\in(0,1]}$. Since

$$u_\varepsilon \in H^1((0,T),H_0^2((0,1))) \subset L^2(X_T),$$

we have equality of the norms

$$||u_\varepsilon||^2_{L^2(X_T)} = ||u_\varepsilon||^2_{L^2((0,T),L^2((0,1)))} \leq (TC'')\varepsilon^{-K''}.$$

Therefore the net of weak solutions $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2(X_T)$-moderate. □

Remark 3.3.3. In the literature, see e.g. [2], a time dependent formulation of moderateness is used. For our problem we could similarly have defined the following moderateness requirement. For each $t \in [0,T]$ the net $(u_\varepsilon(t))_{\varepsilon\in(0,1]}$ is $L^2((0,1))$-moderate. Then existence would follow directly from (3.8). Moreover this bound is uniform for $t \in [0,T]$. By analogy of the Colombeau solution of the Euler-Bernoulli equation in [21], we choose to consider $L^2(X_T)$-moderateness of $(u_\varepsilon)_{\varepsilon\in(0,1]}$ instead.

Now we prove the existence of a very weak solution by constructing regularisations of the distributional coefficients that suffice the requirements of Theorem 3.3.2.

Theorem 3.3.4 (Existence)**.** *Let the initial values $f_1 \in H_0^2((0,1))$ and $f_2 \in L^2((0,1))$. Let the righthand-side and $g \in \mathcal{D}'(X_T)$. Let the coefficients $b \in \mathcal{D}'(X_T)$ and let $c \in L^\infty((0,1))$ such that*

$$0 < c_0 \leq c(x) \leq c_1, \quad \textit{for all } x \in [0,1],$$

for positive constants c_0, c_1. Then there exists a very weak solution to (EB)*,* (bc)*,* (ic)*.*

Proof. Let $(\varphi_\varepsilon)_{\varepsilon\in(0,1]} \subset \mathcal{D}(\mathbb{R}^2)$ be a model delta net. By Theorem 2.9.3 we find the $C^\infty(X_T)$-moderate regularisations

$$b_\varepsilon = b * \varphi_\varepsilon|_{X_T}, g_\varepsilon = g * \varphi_\varepsilon|_{X_T},$$

It follows directly that the net $(b_\varepsilon)_{\varepsilon\in(0,1]}$ is in $C([0,T],L^\infty((0,1)))$ and is $L^\infty(X_T)$-moderate. Similarly $(g_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2(X_T)$-moderate. For c we consider the constant net $(c_\varepsilon)_{\varepsilon\in(0,1]} = (c)_{\varepsilon\in(0,1]}$. These regularisations satisfy the conditions of Theorem 3.3.2. Thus we find and $L^2(X_T)$-moderate net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of weak solutions to (EB$_\varepsilon$), (bc$_\varepsilon$), (ic$_\varepsilon$). We conclude that $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is a very weak solution to (EB), (bc), (ic). □

Remark 3.3.5. Alternatively one can also consider regularisations of c. Let $(\psi_\varepsilon)_{\varepsilon\in(0,1]} \subset \mathcal{D}(\mathbb{R})$ be a positive model delta net such that $\psi_\varepsilon(-x) = \psi_\varepsilon(x)$. Then the regularisation $(c_\varepsilon)_{\varepsilon\in(0,1]} = (c*\psi_\varepsilon|_{(0,1)})_{\varepsilon\in(0,1]}$ is smooth and satisfies the bounds

$$0 < \frac{c_0}{2} \leq c_\varepsilon(x) \leq c_1, \quad \text{for all } x \in [0,1].$$

Now we define uniqueness of the very weak solution.

Definition 3.3.6 (Uniqueness)**.** Let the initial values $f_1 \in H_0^2((0,1))$, $f_2 \in L^2((0,1))$ and the coefficients $b \in \mathcal{D}'(X_T)$, $c \in \mathcal{D}'((0,1))$ and $g \in \mathcal{D}'(X_T)$ be given. Consider the regularisations

- $(b_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\tilde{b}_\varepsilon)_{\varepsilon\in(0,1]}$ regularisations of b in $C([0,T],L^\infty((0,1)))$ and $L^\infty(X_T)$-moderate of log-type, such that $(b_\varepsilon - \tilde{b}_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^\infty(X_T)$-negligible,
- $(c_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\tilde{c}_\varepsilon)_{\varepsilon\in(0,1]}$ regularisations of c that satisfy the bounds

  $$0 < c_0 \leq c_\varepsilon(x) \leq c_1, \quad 0 < \tilde{c}_0 \leq \tilde{c}_\varepsilon(x) \leq \tilde{c}_1, \quad \text{for almost all } x \in [0,1],$$

  for positive constants $c_1, c_2, \tilde{c_1}, \tilde{c_2}$, such that $(c_\varepsilon - \tilde{c}_\varepsilon)_{\varepsilon\in(0,1]}$ is $W^{2,\infty}((0,1))$-negligible,
- $(g_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\tilde{g}_\varepsilon)_{\varepsilon\in(0,1]}$ regularisations of g, such that $(g_\varepsilon - \tilde{g}_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2((0,T),L^2((0,1))$-negligible,

Now let $(u_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ be very weak solutions to (EB), (bc), (ic), by means of the corresponding regularisations of the coefficients. We say that the problem (EB), (bc), (ic) has a unique very weak solution if for all nets $L^2((0,T),H^4((0,1))$-moderate nets $(u_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ we have that $(u_\varepsilon - \tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2(X_T)$-negligible.

Remark 3.3.7. This definition thus only considers uniqueness for $L^2((0,T),H^4((0,1))$-moderate solution $(u_\varepsilon)_{\varepsilon\in(0,1]}$. We would want to strenghten this result by giving minimal conditions on the coefficient regularisations to guarantee this property. Then one can consider uniqueness for the solution nets produces by the sufficiently regular coefficients.

We prove uniqueness in two parts. In Lemma 3.3.8 we apply energy estimate (3.7) in the case of a negligible coefficient $(g_\varepsilon)_{\varepsilon\in(0,1]}$. Then in Theorem 3.3.9 we reduce uniqueness to the case of Lemma 3.3.8.

Lemma 3.3.8. *Let $(b_\varepsilon)_{\varepsilon\in(0,1]} \subset C([0,T],L^\infty((0,1))$ be an $L^\infty(X_T)$-moderate net of log-type and let $(c_\varepsilon)_{\varepsilon\in(0,1]}$ be an $L^\infty((0,1))$-moderate net satisfying the bounds*

$$0 < c_0 \le c_\varepsilon(x) \le c_1, \quad c_0, c_1 \in \mathbb{R}_+,$$

for almost every $x \in [0,1]$, $\varepsilon \in (0,1]$. Assume that the net $(g_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2((0,T),L^2((0,1)))$-negligible. Let $(u_\varepsilon)_{\varepsilon\in(0,1]} \subset L^2((0,T),H_0^2((0,1)))$ be a solution net satisfying (EB$_\varepsilon$), (bc$_\varepsilon$) and initial conditions

$$u_\varepsilon(x,0) = 0, \quad \partial_t u_\varepsilon(x,0) = 0, \quad x \in [0,1].$$

Then $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2(X_T)$-negligible.

Proof. The coefficients $b_\varepsilon, c_\varepsilon$ and g_ε are such that for fixed ε the conditions of Theorem 3.2.2 are satisfied and $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is the unique net of weak solutions. Energy estimate (3.7) thus applies. For the constant F_T^ε we can bound by $L^\infty(X_T)$-moderateness of log-type of $(b_\varepsilon)_{\varepsilon\in(0,1]}$

$$F_T^\varepsilon = \left(||b||_{L^\infty(X_T)} + T + 2\right) \Big/ \min\left(\frac{c_0}{2}, 1\right) \le C \log\frac{1}{\varepsilon},$$

for some $C \ge 0$. We write explicitly the $L^2((0,T),L^2((0,1))$-negligibility of $(g_\varepsilon)_{\varepsilon\in(0,1]}$

$$||g_\varepsilon||_{L^2((0,T),L^2((0,1)))} \le M_q \varepsilon^q, \quad \text{for all } q > 0.$$

Taking into account $||f_1||_{H^2((0,1))} = 0$ and $||f_2||_{L^2((0,1))} = 0$ gives

$$\begin{aligned}
||u_\varepsilon||^2_{L^2(X_T)} &\le \int_0^T \int_0^t ||g_\varepsilon(\tau)||^2_{L^2((0,1))} \exp\left(tF_T^\varepsilon\right) d\tau dt \\
&\le \exp\left(TF_T^\varepsilon\right) \int_0^T \int_0^t ||g_\varepsilon(\tau)||^2_{L^2((0,1)} d\tau dt \\
&\le \exp\left(TC \log\frac{1}{\varepsilon}\right) ||g_\varepsilon||_{L^2((0,T),L^2((0,1)))} \\
&\le M_q \varepsilon^{-TC+q},
\end{aligned}$$

for any $q > 0$. We conclude that $(u_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2(X_T)$-negligible. □

Theorem 3.3.9. *Let $f_1 \in H_0^2((0,1))$, $f_2 \in L^2((0,1))$, $b \in \mathcal{D}'(X_T)$, $g \in \mathcal{D}'(X_T)$ and $c \in L^\infty((0,1))$. Suppose c satisfies the bounds*

$$0 < c_0 \le c_\varepsilon(x) \le c_1, \quad \textit{for all } x \in [0,1],$$

for some positive constants c_0, c_1. Then the problem (EB), (bc), (ic) *has a unique very weak solution in the sense of Definition 3.3.6.*

Proof. Take any solution nets $(u_\varepsilon)_{\varepsilon\in(0,1]}$ and $(\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ and any coefficient nets $(b_\varepsilon)_{\varepsilon\in(0,1]}$, $(\tilde{b}_\varepsilon)_{\varepsilon\in(0,1]}$, $(c_\varepsilon)_{\varepsilon\in(0,1]}$, $(\tilde{c}_\varepsilon)_{\varepsilon\in(0,1]}$, $(g_\varepsilon)_{\varepsilon\in(0,1]}$ and$(\tilde{g}_\varepsilon)_{\varepsilon\in(0,1]}$ that satisfy the conditions from 3.3.6. We need to prove that $(u_\varepsilon - \tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2(X_T)$-negligible. We reduce this theorem to Lemma 3.3.8 for the net $(u_\varepsilon - \tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ and appropriate coeffcients. We have the governing equations

$$\partial_x^2\left(c_\varepsilon(x)\partial_x^2 u_\varepsilon\right) + b_\varepsilon(x,t)\partial_x^2 u_\varepsilon + \partial_t^2 u_\varepsilon = g_\varepsilon(x,t), \tag{EB$_\varepsilon$}$$

$$\partial_x^2\left(\tilde{c}_\varepsilon(x)\partial_x^2 \tilde{u}_\varepsilon\right) + \tilde{b}_\varepsilon(x,t)\partial_x^2 \tilde{u}_\varepsilon + \partial_t^2 \tilde{u}_\varepsilon = \tilde{g}_\varepsilon(x,t), \tag{$\widetilde{EB}_\varepsilon$}$$

Substracting equation $(\widetilde{EB}_\varepsilon)$ from $(\mathrm{EB}_\varepsilon)$ and rearranging gives

$$\partial_x^2\left(c_\varepsilon\partial_x^2(u_\varepsilon-\tilde{u}_\varepsilon)\right)+b_\varepsilon\partial_x^2(u_\varepsilon-\tilde{u}_\varepsilon)+\partial_t^2(u_\varepsilon-\tilde{u}_\varepsilon)=(g_\varepsilon-\tilde{g}_\varepsilon)+(\tilde{b}_\varepsilon-b_\varepsilon)\partial_x^2\tilde{u}_\varepsilon+\partial_x^2((\tilde{c}_\varepsilon-c_\varepsilon)\partial_x^2\tilde{u}_\varepsilon). \tag{3.9}$$

We write the righthandside of (3.9) as $h_\varepsilon(x,t)$. The net $(h_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2((0,T),L^2((0,1)))$-negligible since each of its terms is. We verify that. The difference $(g_\varepsilon-\tilde{g}_\varepsilon)_{\varepsilon\in(0,1]}$ is negligible by assumption. The term $(\tilde{b}_\varepsilon-b_\varepsilon)\partial_x^2\tilde{u}_\varepsilon$ is a product of an $L^\infty(X_T)$ function with an $L^2(X_T)$. Then the Holder's inequality 1.4 gives the bound

$$||(\tilde{b}_\varepsilon-b_\varepsilon)\partial_x^2\tilde{u}_\varepsilon||_{L^2(X_T)}\leq||(\tilde{b}_\varepsilon-b_\varepsilon)||_{L^\infty(X_T}||\partial_x^2\tilde{u}_\varepsilon||_{L^2(X_T)}.$$

The product of the negligible net $(||(\tilde{b}_\varepsilon-b_\varepsilon)||_{L^\infty(X_T)})_{\varepsilon\in(0,1]}$ and the moderate net $(||\partial_x^2\tilde{u}_\varepsilon||_{L^2})_{\varepsilon\in(0,1]}$ is again negligible. For the third term $\partial_x^2\left((\tilde{c}_\varepsilon-c_\varepsilon)\partial_x^2\tilde{u}_\varepsilon\right)$, it is sufficient that $(\tilde{c}_\varepsilon-c_\varepsilon)\partial_x^2\tilde{u}_\varepsilon$ is $L^2((0,T),H^2((0,1)))$-negligible. Since

$$(\tilde{c}_\varepsilon-c_\varepsilon)_{\varepsilon\in(0,1]}\subset C^\infty([0,T],W^{2,\infty}((0,1))),$$

and

$$(\partial_x^2\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}\subset L^2((0,T),H^2((0,1))),$$

the Sobolev product $(\tilde{c}_\varepsilon-c_\varepsilon)\partial_x^2\tilde{u}_\varepsilon$, is again in $L^2((0,T),H^2((0,1)))$. With the the Leibniz rule for the Sobolev product and then applying the triangle inequality and Holder's inequality we get

$$\begin{aligned}
||\partial_x^2\left((\tilde{c}_\varepsilon-c_\varepsilon)\partial_x^2\tilde{u}_\varepsilon\right)||_{L^2(X_T)}&=||\partial_x^2\left(\tilde{c}_\varepsilon-c_\varepsilon\right)\partial_x^2\tilde{u}_\varepsilon+\partial_x\left(\tilde{c}_\varepsilon-c_\varepsilon\right)\partial_x^3\tilde{u}_\varepsilon+(\tilde{c}_\varepsilon-c_\varepsilon)\partial_x^4\tilde{u}_\varepsilon||_{L^2(X_T)} && (3.10)\\
&\leq||\partial_x^2(\tilde{c}_\varepsilon-c_\varepsilon)||_{L^\infty(X_T)}||\partial_x^2\tilde{u}_\varepsilon||_{L^2(X_T)}+||\partial_x(\tilde{c}_\varepsilon-c_\varepsilon)||_{L^\infty(X_T)} && (3.11)\\
&\quad||\partial_x^3\tilde{u}_\varepsilon||_{L^2(X_T)}+||\tilde{c}_\varepsilon-c_\varepsilon||_{L^\infty(X_T)}||\partial_x^4\tilde{u}_\varepsilon||_{L^2(X_T)} && (3.12)\\
&\leq 3||\tilde{c}_\varepsilon-c_\varepsilon||_{W^{2,\infty}}||\partial_x^2\tilde{u}||_{L^2((0,T),H^2((0,1))} && (3.13)\\
&=3||\tilde{c}_\varepsilon-c_\varepsilon||_{W^{2,\infty}}||\tilde{u}||_{L^2((0,T),H^4((0,1))}. && (3.14)
\end{aligned}$$

Since $(c_\varepsilon)_{\varepsilon\in(0,1]}$ is $W^{2,\infty}$-moderate and $(\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2((0,T),H^4((0,1))$-moderate the bound (3.14) is negligible. Therefore the term $\partial_x^2((\tilde{c}_\varepsilon-c_\varepsilon)\partial_x^2\tilde{u}_\varepsilon)$ is $L^2(X_T)$-negligible. We conclude that $(h_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2((0,T),L^2((0,1))$-negligible. Now $u_\varepsilon-\tilde{u}_\varepsilon$ satisfies the boundary conditions (bc) and the initial conditions

$$(u_\varepsilon-\tilde{u}_\varepsilon)(x,0)=0,\quad\text{and}\quad\partial_t(u_\varepsilon-\tilde{u}_\varepsilon)(x,0)=0.$$

Together with (3.9), the conditions of Theorem 3.3.8 are fulfilled for the solution net $(u_\varepsilon-\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ with coefficients c_ε and b_ε and the righthandside h_ε. Therefore $(u_\varepsilon-\tilde{u}_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^2(X_T)$-negligible. □

The question of consistency of the very weak solution was not yet considered. Next we analyse numerically the solutions to the Euler-Bernoulli equation with distributional forces.

3.4 Numerical analysis

In this section we study the solutions of regularisations of the Euler-Bernoulli equation with distributional forces. We will solve the regularised equation numerically for small ε. We use the finite element method (FEM) introduced in Section 1.5. First we construct the finite element solution of the Euler-Bernoulli equation. Then we analyse the numerical results for several examples of distributional coefficients.

3.4.1 Finite element solution

We construct the Galerkin approximation (1.19) of the variational problem (3.5) with boundary conditions (3.6). We look for a solution in the space

$$V=\left\{f\in H^2((0,1)):f(0)=f(1)=0,\,\partial_xf(0)=\partial_xf(1)=0\right\}.$$

To implement boundary conditions we choose the Galerkin approximation $V_h=V\cap V_h^{1,3}$ to implement boundary conditions. That is

$$V_h=\left\{u\in V_h^{1,3}:\,u(0)=u(1)=0,\,u'(0)=u'(1)=0\right\}=\left\{\sum_{j=1}^{n-1}\alpha_j\varphi_j+\sum_{j=1}^{n-1}\beta_j\psi_j,\,\alpha_j,\beta_j\in\mathbb{R}\right\}.$$

where the basis functions φ_i, ψ_i of $V_h^{1,3}$ are given by

$$\varphi_0(x) = \begin{cases} \frac{1}{h^2}(x-h)^2(1+\frac{2x}{h}), & \text{if } 0 \le x \le h, \\ 0, & \text{otherwise.} \end{cases}$$

$$\varphi_n(x) = \begin{cases} \frac{1}{h^2}(x-1+h)^2(1+\frac{2}{h}(1-x)), & \text{if } (n-1)h \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $j = 1, \ldots, n$

$$\varphi_j(x) = \begin{cases} \frac{2}{h^2}(x-(j-1)h)^2(1+\frac{2}{h}(jh-x)), & \text{if } (j-1)h \le x \le jh, \\ \frac{2}{h^2}(x-(j+1)h)^2(1-\frac{2}{h}(jh-x)), & \text{if } jh \le x \le (j+1)h, \\ 0, & \text{otherwise,} \end{cases}$$

$$\psi_0(x) = \begin{cases} \frac{1}{h^2}(x-h)^2 x, & \text{if } 0 \le x \le h, \\ 0, & \text{otherwise,} \end{cases}$$

$$\psi_n(x) = \begin{cases} \frac{1}{h^2}(x-1+h)^2(x-1), & \text{if } (n-1)h \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $j = 1, \ldots, n$

$$\psi_j(x) = \begin{cases} \frac{1}{h^2}(x-(j-1)h)^2(x-jh), & \text{if } (j-1)h \le x \le jh, \\ \frac{1}{h^2}(x-(j+1)h)^2(x-jh), & \text{if } jh \le x \le (j+1)h, \\ 0, & \text{otherwise.} \end{cases}$$

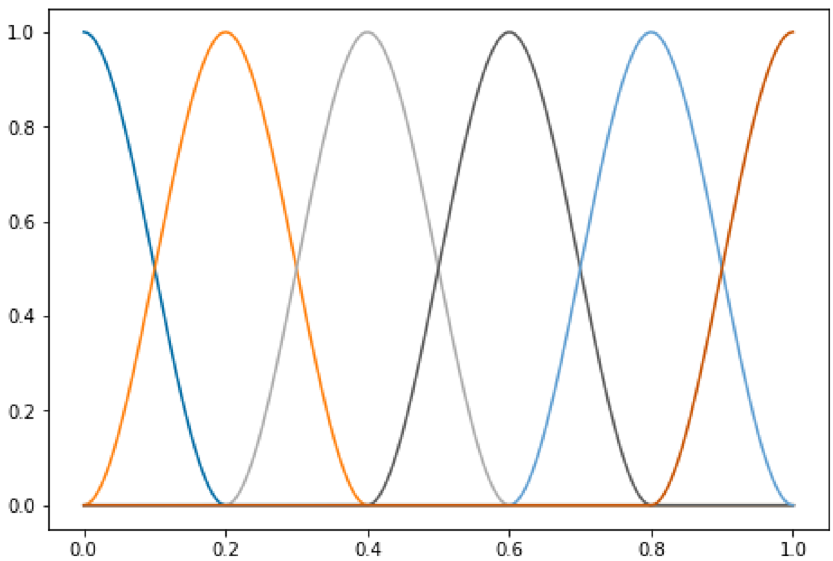


Figure 3.1: Basis functions of $V_h^{1,3}$ on $[0,1]$, $n = 5$, $h = 0.2$. Left to right: $\varphi_0, \ldots, \varphi_5$

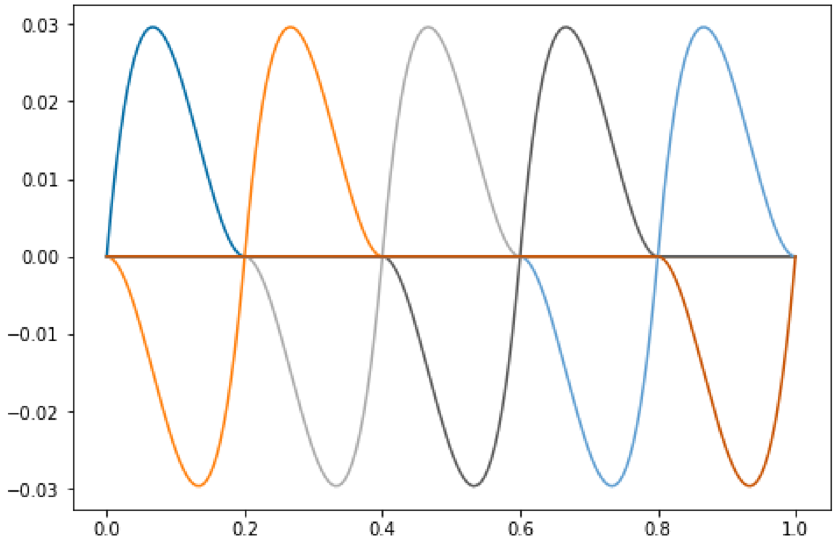


Figure 3.2: Basis functions of $V_h^{1,3}$ on $[0,1]$, $n=10$, $h=0.2$. Left to right: $\psi_0,\dots,\psi_5$

The basis functions have the following properties. For $j=0,\dots,n$,

$$\begin{aligned}\operatorname{supp}\varphi_j = \operatorname{supp}\psi_j &= [(j-1)h,(j+1)h]\cap[0,1],\\ \varphi_j(jh) &= 1,\\ \psi_j'(jh) &= 1.\end{aligned}$$

Moreover, the second order weak derivatives $\varphi''(x)$ and $\psi''(x)$ are piecewise linear and thus in $L^2((0,1))$. Thus $V_h^{1,3}\subset H^2((0,1))$ for any $n\in\mathbb{N}$. A function $u_h\in L^2((0,T),V_h)$ can be written in terms of basis functions

$$u_h(x,t)=\sum_{j=1}^{n-1}\alpha_j(t)\varphi_j(x)+\sum_{j=1}^{n-1}\beta_j(t)\psi_j(x), \tag{3.15}$$

for time dependent coefficients $\alpha_j(t),\beta_j(t)\in L^2((0,T))$, $j=1,\dots,n-1$. For fixed t, the finite element solution u_h then satisfies

$$\int_0^1 \partial_t^2 u_h(x,t)v(x)dx+\int_0^1 c(x)\partial_x^2 u_h(x,t)\partial_x^2 v(x)dx+\int_0^1 b(x,t)\partial_x^2 u_h(x,t)v(x)dx=\int_0^1 g(x,t)v(x)dx, \tag{3.16}$$

for all $v\in V_h$. By taking linear combinations, one sees that it is sufficient that (3.16) is satisfied only for basis functions $v=\varphi_i,\psi_i, i=1,\dots,n-1$. We will write the system of linear equations (3.16) in matrix form.

$$a_{i,j} = \begin{cases} \int_0^1 \partial_x^2\varphi_j(x)\partial_x^2\varphi_i(x)dx, & \text{if } 1 \le i,j \le n-1 \\ \int_0^1 \partial_x^2\psi_{j-(n-1)}(x)\partial_x^2\varphi_i(x)dx, & \text{if } 1 \le i \le n-1, 1 \le j-(n-1) \le n-1, \\ \int_0^1 \partial_x^2\varphi_j(x)\partial_x^2\psi_{i-(n-1)}(x)dx, & \text{if } 1 \le i-(n-1) \le n-1, 1 \le j \le n-1, \\ \int_0^1 \partial_x^2\psi_{j-(n-1)}(x)\partial_x^2\psi_{i-(n-1)}(x)dx, & \text{if } 1 \le k-(n-1) \le n-1, k=i,j. \end{cases} \tag{3.17}$$

$$c_{i,j} = \begin{cases} \int_0^1 c(x)\partial_x^2\varphi_j(x)\partial_x^2\varphi_i(x)dx, & \text{if } 1 \le i,j \le n-1 \\ \int_0^1 c(x)\partial_x^2\psi_{j-(n-1)}(x)\partial_x^2\varphi_i(x)dx, & \text{if } 1 \le i \le n-1, 1 \le j-(n-1) \le n-1, \\ \int_0^1 c(x)\partial_x^2\varphi_j(x)\partial_x^2\psi_{i-(n-1)}(x)dx, & \text{if } 1 \le i-(n-1) \le n-1, 1 \le j \le n-1, \\ \int_0^1 c(x)\partial_x^2\psi_{j-(n-1)}(x)\partial_x^2\psi_{i-(n-1)}(x)dx, & \text{if } 1 \le k-(n-1) \le n-1, k=i,j. \end{cases} \tag{3.18}$$

$$b_{i,j}(t) = \begin{cases} \int_0^1 b(x,t)\partial_x^2\varphi_j(x)\varphi_i(x)dx, & \text{if } 1 \le i,j \le n-1 \\ \int_0^1 b(x,t)\partial_x^2\psi_{j-(n-1)}(x)\varphi_i(x)dx, & \text{if } 1 \le i \le n-1, 1 \le j-(n-1) \le n-1, \\ \int_0^1 b(x,t)\partial_x^2\varphi_j(x)\psi_{i-(n-1)}(x)dx, & \text{if } 1 \le i-(n-1) \le n-1, 1 \le j \le n-1, \\ \int_0^1 b(x,t)\partial_x^2\psi_{j-(n-1)}(x)\psi_{i-(n-1)}(x)dx, & \text{if } 1 \le k-(n-1) \le n-1, k=i,j. \end{cases} \tag{3.19}$$

$$g_i(t) = \begin{cases} \int_0^1 g(x,t)\varphi_i(x)dx, & \text{if } 1 \le i \le n-1 \\ \int_0^1 g(x,t)\psi_{i-(n-1)}(x)dx, & \text{if } 1 \le i-(n-1) \le n-1. \end{cases} \tag{3.20}$$

Let A, $B(t)$ and C be the matrices of dimension $(2n-2)\times(2n-2)$ with elements $a_{i,j}, b_{i,j}(t)$ and $c_{i,j}$ respectively and let $G(t)$ be the column vector of dimension $2n-2$ with elements $g_i(t)$.

It should be noted that in a numerical setting, the integrals in (3.17), (3.18), (3.19) and (3.20) might need to be numerically computed or approximated. Especially the coefficients $c_{i,j}$ turned out to be hard to approximate numerically exactly. Define the coefficient vector

$$U(t) = \left\{\alpha_1, \dots \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}\right\}^T.$$

Then equation (3.16) reads

$$A\partial_t^2 U(t) + (B(t)+C)U(t) = G(t). \tag{3.21}$$

To solve the system of differential equations (3.21), we employ the finite difference method. Split the time interval $[0,T]$ into m equal parts with midpoints $t_k = kh_t, k = 0, \dots m, h_t = \frac{1}{m}$. We consider equation (3.21) only in the points $t = t_k$. Next we approximate the second order time derivative by the backward difference operator

$$\partial_t^2 U(t_k) \approx \frac{U(t_{k-2}) - 2U(t_{k-1}) + U(t_k)}{h_t^2}, \quad k = 0, \dots, m.$$

Then (3.21) becomes

$$A\frac{U(t_{k-2}) - 2U(t_{k-1}) + U(t_k)}{h_t^2} + (B(t_k)+C)U(t_k) = G(t_k), \quad k = 0, \dots, m.$$

This gives an implicit scheme for the coefficients

$$(A + h_t^2(B(t_k)+C))U(t_k) = h_t^2 G(t_k) - A(U(t_{k-2}) - 2U(t_{k-1})), \quad k = 0, \dots, m. \tag{3.22}$$

One can solve for $U(t_k)$ by multiplying (3.22) by the inverse matrix of $(A + h_t^2(B(t_k)+C))$. From the coefficients $U(t_k)$ we construct the numerical solution by by equation (3.15).

By itself the central difference operator

$$\partial_t^2 U(t_{k-1}) \approx \frac{U(t_{k-1}) - 2U(t_{k-1}) + U(t_k)}{h_t^2}, \quad k = 0, \dots, m,$$

is a better approximating for the second order time derivative than the backward difference operator. This however makes an explicit scheme, which is less numerically stable than the implicit scheme created by the backward difference operator.

Next we implement the initial conditions

$$u(x,0) = f_1(x), \quad \partial_t u(x,0) = f_2(x).$$

We approximate the intial state to $u_h(x,0)$ by means of the interpolation operator Π_h. We put

$$\alpha_j(t_0) = f_1(jh), \quad \beta_j(t_0) = \partial_x f_1(jh), \quad j = 1, \dots, n-1,$$

then

$$\Pi_h f_1(x) = \sum_{j=1}^{n-1} \alpha_j(t_0)\varphi_j(x) + \sum_{j=1}^{n-1} \beta_j(t_0)\psi_j(x).$$

Given that f_1 is sufficiently smooth, $\Pi_h f_1$ will be a good approximation of f_1. For the initial velocity, we approximate

$$\Pi_h f_2(x) \approx f_2(x) \approx \frac{u(x,t_0) - u(x,t_{-1})}{h_t} \approx \frac{\Pi_h u(x,0) - u(x,t_{-1})}{h_t},$$

therefore we set

$$U(t_{-1}) = \Pi_h f_1(x) + h_t \Pi_h f_2(x).$$

3.4.2 Regularisation

We describe the numerical procedure for the regularisation of the coefficients. We consider the standard mollifier in $\mathbb{R}^n$

$$\varphi(x) = \begin{cases} A\exp\left(\dfrac{1}{1-|x|^2}\right), & \text{if } |x| < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.23}$$

The constant A is chosen such that

$$\int_{\mathbb{R}^n} \varphi(x)dx = 1.$$

Then we have the model delta net $\varphi_\varepsilon(x) = \dfrac{1}{\varepsilon^n}\varphi\left(\dfrac{x}{\varepsilon}\right)$. For our experiments, we will fix ε to be a small number. So fix one corresponding φ_ε. Write f_ε for $f * \varphi_\varepsilon$. If $f \in \mathcal{D}'(X_T)$, then $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} f_\varepsilon \subseteq X_T + \varepsilon$. This has the consequence that f_ε is small at the the boundary ∂X_T whenever f is bounded around ∂X_T. This could lead to instability of the numerical solutions. We describe an improved regularisation procedure. Suppose that f has uniquely determined values $f(x)$ on the boundary ∂X_T. We extend the domain of f to $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$, define x_p as the closest point to x in X_T. We can then define an extension, constant from the boundary

$$\bar{f}(x) = \begin{cases} f(x), & x \in X_T, \\ f(x_p), & x \notin X_T. \end{cases}$$

Now we consider the regularisation $\bar{f}_\varepsilon$. Then $\bar{f}_\varepsilon$ will be approximately the value near the boundary of f.

Suppose that we consider a constant bending stiffness $c(x) = 1$ on $[0,1]$. The naive regularisation we mentioned is given in figure (3.3), while the regularisation of the extension is $\bar{c}_\varepsilon(x) = 1$, independent of ε. It is clear that this procedure results in more smooth functions at the boundary.

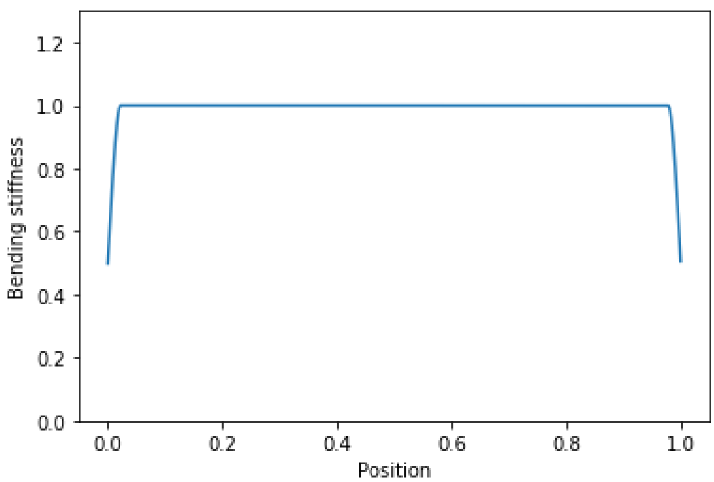


Figure 3.3: The regularisation $c_\varepsilon(x)$ of $c(x) = 1$ for $\varepsilon = 0.05$.

3.4.3 Numerical experiments

We want to use numerical analysis to gain insight into the Euler-Bernoulli beam with discontinuous cross-section and singular forces. Our approach is through regularisation of the singular forces. We use the finite element method of Section 3.4.1 to numerically approximate the solutions.

Constant coefficients

We start with a basic case to see what can be expected from a solution to the Euler-Bernoulli beam equation. We consider one of the simplest non-trivial cases of the Euler-Bernoulli equation.

$$\begin{aligned} f_1(x) &= 0, \\ f_2(x) &= 0, \\ b(x,t) &= 1, \\ g(x,t) &= 1, \\ c(x) &= 1. \end{aligned} \tag{3.24}$$

The boundary value problem we solve is then

$$\begin{aligned} \partial_t^2 u(x,t) + \partial_x^4 u(x,t) + \partial_x^2 u(x,t) &= 1, \\ u(x,0) = 0, \quad \partial_t u(x,t) &= 0. \end{aligned} \tag{3.25}$$

We use the finite element method of Section 3.4.1 to numerically approximate the solution $u(x,t)$. We visualise $u(x,t)$ in a 3d-plot. Figure 3.4 presents the solution to (3.25).

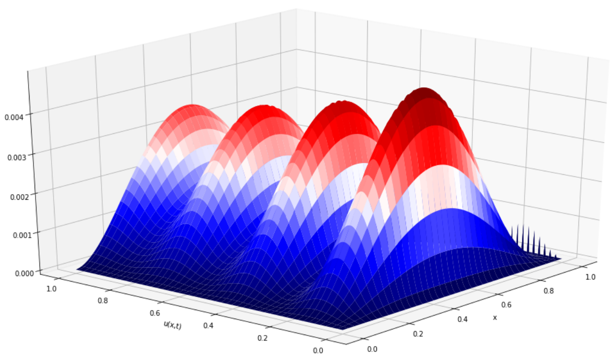


Figure 3.4: The solution $u(x,t)$ to (3.25), the Euler-Bernoulli equation with constant coefficients. The beam vibrates and converges to an equilibrium state. Numerical settings: $T = 1$, $n = 200$, $m = 200$.

The solution $u(x,t)$ can be interpreted as the beam vibrating around and converging to an equilibrium state. The vibration can be seen more easily if we track a single point in time, see Figure 3.5.

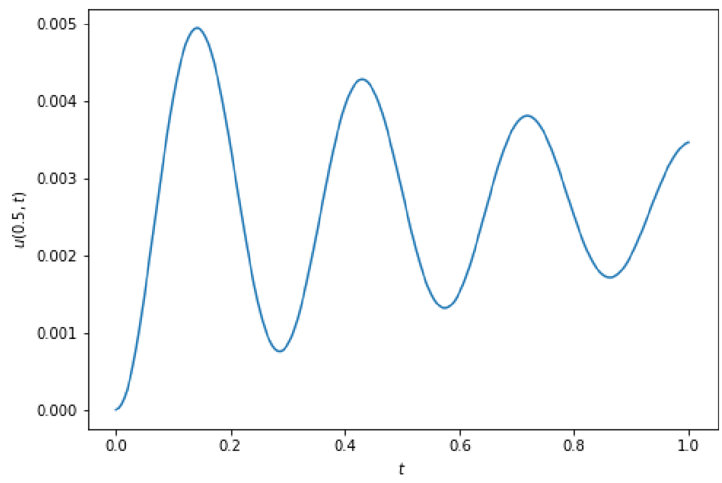


Figure 3.5: The function $u(0.5, t)$, where $u(x, t)$ is solution to (3.25), the Euler-Bernoulli equation with constant coefficients. The point on the beam follows the trajectory of a damped vibration. Numerical settings: $T = 1$, $n = 200$, $m = 200$.

Remark 3.4.1. We mention a small numerical anomaly in our simulations which can be seen in Figure 3.4 for $x \approx 1$. This numerical error exists in all the results we present in this section. The anomaly does not significantly impact the whole numerical solution of the Euler-Bernoulli beam. The anomaly is most likely the result of the accumulation of numerical errors at a matrix inversion.

Segmented beam

We want to consider a beam with discontinuous cross-section. The bending stiffness c is then of Heaviside type

$$c(x) = EI_1 + (EI_2 - EI_1)H(x - a), \quad a \in (0, 1),\ E, I_1, I_2 \in \mathbb{R}^+. \tag{3.26}$$

Suppose b and g satisfy (3.24) and where c is given by (3.26) with $a = 0.5, EI_1 = 1$ and $EI_2 = 5$. The coresponding regularised boundary value problem is

$$\begin{aligned} \partial_t^2 u(x,t) + \partial_x^2 \Big(1 + 4H(x - 0.5)\partial_x^2 u(x,t)\Big) + \partial_x^2 u(x,t) = 1, \\ u(x, 0) = 0, \quad \partial_t u(x,t) = 0. \end{aligned} \tag{3.27}$$

The solution $u(x, t)$ to (3.27) is presented in Figure 3.6. Again the beam vibrates and converges to an equilibrium. We consider the shape of the equilibrium state further by Figure 3.7 3.7. The physical interpretation of the bending stiffness c can be seen as follows. The beam part ($x \geq 0.5$) with higher bending stiffness is less bent then the beam part ($x \leq 0.5$) with a lower bending stiffness.

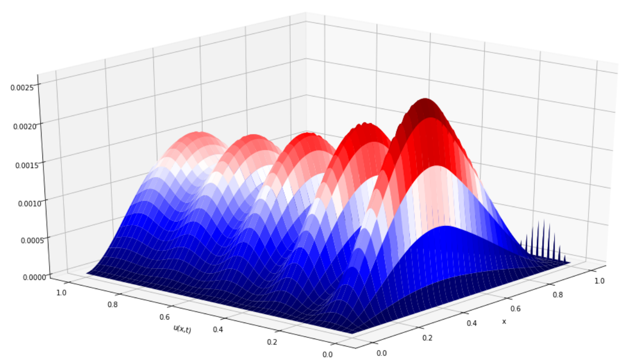


Figure 3.6: The solution $u(x, t)$ to (3.27), the Euler-Bernoulli equation with discontinuous cross-section. The beam vibrates and converges to an equilibrium state. The part of the beam ($x \geq 0.5$) with high bending stiffness $c(x)$ is less bent than the part of the beam ($x \leq 0.5$) with lower bending stiffness. The two beam parts meet at the point of discontinuity Numerical settings: $T = 1$, $n = 200$, $m = 200$.

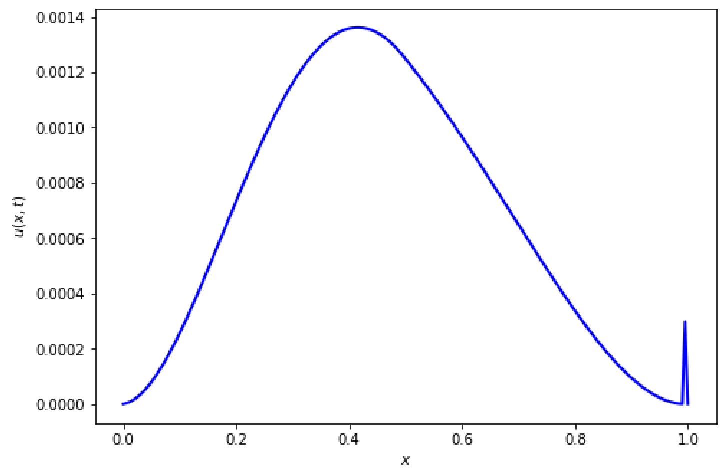


Figure 3.7: The function $u(x, 1)$, where $u(x, t)$ is the solution to (3.27), the Euler-Bernoulli equation with discontinuous cross-section. The plot shows the beam at a time close to the equilibrium state. The part of the beam ($x \geq 0.5$) with high bending stiffness $c(x)$ is less bent than the part of the beam with lower bending stiffness ($x \leq 0.5$). Numerical settings: $T = 1$, $n = 200$, $m = 200$.

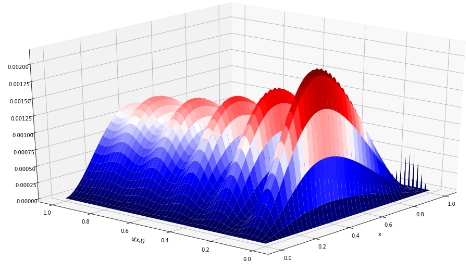
Now we investigate the behaviour of regularisation of $c(x)$ of Heaviside type. The regularisation c_ε of c is

$$c_\varepsilon(x) = (c * \varphi_\varepsilon)(x),$$

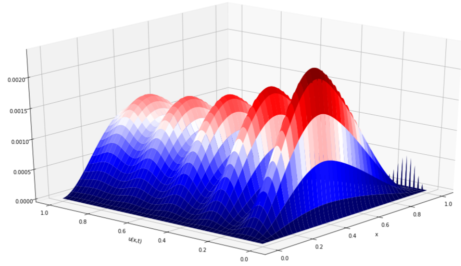
with φ_ε the standard mollifier (3.23). The boundary value problem is

$$\begin{aligned} \partial_t^2 u(x,t) + \partial_x^2 \left(c_\varepsilon(x) \partial_x^2 u(x,t) \right) + \partial_x^2 u(x,t) = 1, \\ u(x,0) = 0, \quad \partial_t u(x,t) = 0. \end{aligned} \tag{3.28}$$

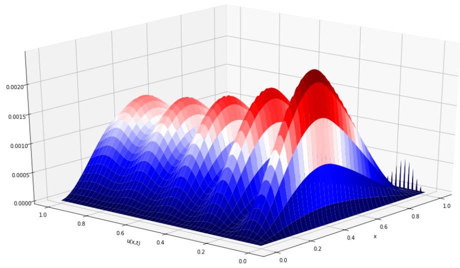
In Figure 3.8 the numerical solution to (3.28) for $\varepsilon = 0.2, 0.1, 0.05$ is presented. On the left graph, around the piont $x = 0.5$ the beam bends more gradually than for the discontinuous case, since c_ε changes less abrubtly than c. Therefore the stress force applies more evenly. In the right and bottom graphs, the result looks more comparable to the situation without regularisation in Figure 3.6. The smoothing happens on the scale $[-\varepsilon, \varepsilon]$ and is thus hard to notice. We chose the values for ε such that the width of the finite element discretisation $h = 0.005$ (since $n = 200$) is sufficiently smaller than ε. This guarantees that the regularisation procedure is not lost to the numerical approximation.



(a) $\varepsilon = 0.2$



(b) $\varepsilon = 0.1$



(c) $\varepsilon = 0.05$

Figure 3.8: The solution $u(x,t)$ to (3.28) for $\varepsilon = 0.2, 0.1, 0.05$, the regularised Euler-Bernoulli equation with discontinuous cross-section. Numerical settings: $T = 1$, $n = 200$, $m = 200$.

Point force

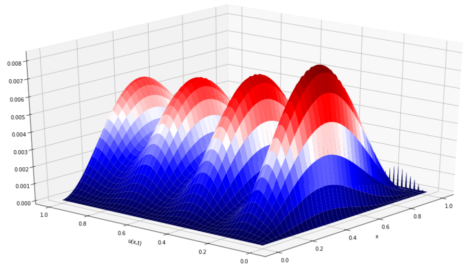
We consider a stationary point masss at point $x = a$ modelled by the distributional force $g(x,t) = \delta(x-a)$. Let other coefficients be constant as in (3.24). We analyse the regularisation of the corresponding boundary value problem. First we regularise g as

$$\begin{aligned} g_\varepsilon(x,t) &= (\delta(y-a) * \varphi_\varepsilon(y,\tau))(x,t), \\ &= \int_{-\infty}^{+\infty} \varphi_\varepsilon(x-a,\tau) d\tau. \end{aligned} \tag{3.29}$$

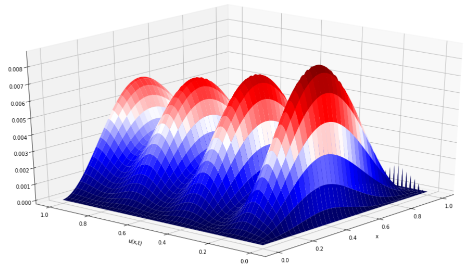
with φ_ε the standard mollifier (3.23) in $\mathbb{R}^2$. We solve the regularised equation

$$\begin{aligned} \partial_t^2 u(x,t) + \partial_x^4 u(x,t) + \partial_t^2 u(x,t) &= g_\varepsilon(x,t), \\ u(x,0) = 0, \quad \partial_t u(x,0) &= 0. \end{aligned} \tag{3.30}$$

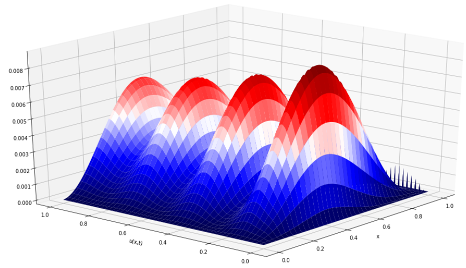
We choose $a = 0.4$ so that the assymmetry may reveal more of the behaviour of the point force. For $\varepsilon = 0.2, 0.1, 0.05$ the solution to (3.30) is plotted in Figure 3.9. Compared to the constant force, Figure 3.4, the point force behaves very similarly. This is due to the bending coefficient $\partial_x^2 c(x) \partial_x^2 u(x,t)$, which makes the beam bend as a whole.



(a) $\varepsilon = 0.2$



(b) $\varepsilon = 0.1$



(c) $\varepsilon = 0.05$

Figure 3.9: The solution $u(x,t)$ to (3.30) for $\varepsilon = 0.2, 0.10.05$, regularised Euler-Bernoulli equation with point force. Numerical settings $T = 1.$, $n = 200$, $m = 200$.

Segmented beam with crack

A crack in the beam at the point $x = a$ is modeled by $b(x,t) = b_0\delta(x-a)$. This models the fact that the stress induced by the axial force accumulates at a crack at the position $x = a$. We consider the regularised boundary value problem. We construct an L^∞ log-type moderate regularisation of b. Convolution of b by the standard mollfier (3.23) gives similarly to (3.29)

$$\varphi_\varepsilon * b(x,t) = b_0 \int_{-\infty}^{+\infty} \varphi_\varepsilon(x-a,\tau)d\tau = \frac{b_0}{\varepsilon^2}\int_{-\infty}^{+\infty} \varphi((x-a)/\varepsilon, \tau/\varepsilon)d\tau = \frac{b_0}{\varepsilon}\int_{-\infty}^{+\infty} \varphi((x-a)/\varepsilon, \tau)d\tau.$$

Thus we have the $L^\infty(X_T)$-moderateness bound

$$||\varphi_\varepsilon * b(x,t)||_{L^\infty(X_T)} \leq C\varepsilon^{-N},$$

with

$$C = b_0 \sup_{x\in(0,1)} \int_{-\infty}^{+\infty} \varphi((x-a)/\varepsilon, \tau)d\tau,$$

and $N = 1$. Therefore we can use (2.28) for $N = 1$. That is

$$\lambda_\varepsilon = \left(\log\frac{1}{\varepsilon}\right)^{-1}$$

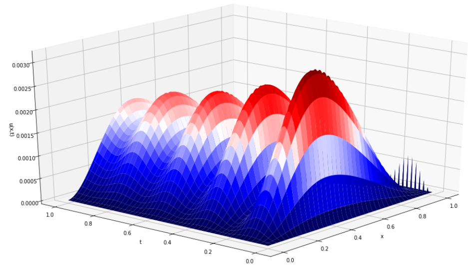
and the reparametrised regularisation

$$b_\varepsilon(x,t) = (\varphi_{\lambda_\varepsilon}(\xi,\tau) * b(\xi,\tau))(x,t) = \frac{b_0}{\lambda_\varepsilon}\int_{-\infty}^{+\infty} \varphi((x-a)/\lambda_\varepsilon, \tau)d\tau,$$

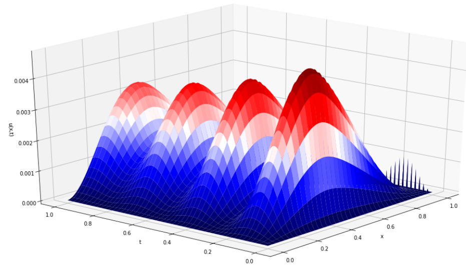
which is an $L^\infty(X_T)$-moderate regularisation of log-type of b. The speed of convergence of the regularisation of b is meaningless if b is the only coeffcient regularised. If corresponding net of solutions $(u_\varepsilon)_{\varepsilon\in(0,1]}$ has a limit u, then any reparametrisation will have the same limit. This why we choose to regularise a second coeffcient in this example, the segmented beam $c(x) = 1 + 4H(x-0.5)$ in this example. Setting $g = 1$ constant, we thus consider the following regularised problem.

$$\begin{aligned} \partial_t^2 u(x,t) + \partial_x^2(c_\varepsilon(x)\partial_x^2 u(x,t)) + b_\varepsilon(x,t)\partial_x^2 u(x,t) &= 1, \\ u(x,0) = 0, \quad \partial_t u(x,t) &= 0. \end{aligned} \tag{3.31}$$

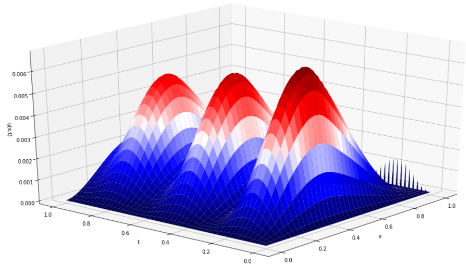
We pick the case $a = 0.5$, $b_0 = 20$. The numerical solution for $\varepsilon = 0.2, 0.1, 0.05$ is given in Figure (3.10).



(a) $\varepsilon = 0.2$



(b) $\varepsilon = 0.1$



(c) $\varepsilon = 0.05$

Figure 3.10: The solution $u(x,t)$ to (3.31) for $\varepsilon = 0.2, 0.1, 0.05$, the Euler-Bernoulli beam with discontinuous cross-section and crack $a = 0.5$ of strength $b_0 = 20$. Numerical settings $T = 4$, $n = 200$, $m = 200$.

Figure (3.10) shows that the beam converges again to an equilibrium state. The effect of the beam crack is minimal since the corresponding regularisation parameter for b at $\varepsilon = 0.05$ is $\lambda_\varepsilon \approx 0.33$. For smaller epsilon the beam vibrates slower. However since the regularisation parapeter $\lambda_\varepsilon \geq 0.33$ the regularisation b_{λ_ε} is not a good approximation of b.

Our numerical results show that a numerical solution to the Euler-Bernoulli beam equation with distributional forces and discontinuous cross-section can be found through regularisation. Depending on the specific distributional coefficients, the results of the regularised equation are physically interpretable. Further numerical analysis can be performed about time-dependent forces and non-zero initial conditions. Also more exotic distributions, e.g. a point force of derivative delta type $g(x,t) = \partial_x \delta(x)$, can be considered. The experiments can also be performed for non-smooth regularisations of the coefficients as is the case in the very weak solution. The moderateness of the solution net $(u_\varepsilon)_{\varepsilon \in (0,1]}$ can be numerically verified.

Chapter 4

Conclusion

In this thesis we discussed the muliplication problem for distributions. We discussed several definitions and methods for multiplication maps. The Schwartz product is a standard definition in distribution theory and defines a product between a distribution and a smooth function. However even the Schwartz product is a continuous map $\mathcal{D}'(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$. Then we localised the Schwartz prodcuct to a product for distributions with disjoint singular support. The localisation procedure is applicable to all further products. The duality method can produce a wide range of products that can be used in specific applications. It also gives us a product for Sobolev spaces. The Fourier product generalises the exchange formula for the $\mathcal{S}'$-convolution. The restriction to $\mathcal{S}'$convolutions optimises the properties against the generality of the product. The strict product and the model product are defined through regularisation of the factors. The limit of the regularised products is then taken as the distributional product, if the limit is equal for all regularisations by strict and model delta nets respectively. There are different products depending on how the factors are regularised. These products allow for a very general multiplication of distributions. They can still be extended by considering smaller classes of delta nets. Model product (model4) is especially important since it captures the multiplication of distributions by Colombeau algebra, see Theorem 2.9.5.

The distributional products we discussed is not a complete discussion of the multiplication problem. There are many other methods proposed in the literature, e.g. the parameter product, the Tillman product and the neutrix product. We discussed extrinsic multiplciation in associative commutative algebras of generalised distributions. Schwartz's impossibility result says that we can't have an extrinsic product that keeps the pointwise multiplication of continuous functions. The Colombeau algebra of generalised functions succeeds in keeping the pointwise product for smooth functions. We briefly introduced the special Colombeau algebra $\mathcal{G}^s$.

Next we introduced the very weak solution concept. Through a simple example we explained the methods to define existence and uniqueness of the very weak solution to a partial differential equation. We did however not discuss how one should approach proving consistency with classical results. This is a key part to any method that uses regularisation.

In the third chapter we defined a very weak solution to the clamped Euler-Bernoulli with discontinuous cross-section and distributional forces. We proved existence of the very weak solution. Then we proved uniqueness of the very weak solution provided that the very weak solution is $L^2((0,T), H^4((0,1))$-moderate. If sufficiently many derivatives of the coefficient regularisations are moderate, this condition is fulfilled. However a minimal condition where this is the case is not given. This is a topic for further research. Also consistency of the very weak solution with classical results was not yet investigated. In the numerical analysis we analysed different cases of the Euler-Bernoulli beam with distributional forces. We only considered a few time-independent coefficients. To extend this numerical analysis one can consider time-dependent coefficient and more exotic distribution coefficients. Also one can analyse the moderateness bounds of the regularised solution nets.

Appendix A

Nederlandstalige samenvatting.

Deze master thesis bevat drie hoodstukken. Het eerste hoofstuk dient als een inleidend hoofdstuk. We beschrijven standaard concepten en nuttige resultaten uit distributietheorie, functionaalanalyse, partiële differentiaalvergelijkingen en numerieke analyse.

Hoofstuk twee gaat over het vermenigvuldigingsprobleem van distributies. We baseren ons op het boek 'Multiplication of distributions and applications to partial differential equations' [30] van Oberguggenberger. Het is geweten dat sommige producten van distributies, geen distributie kunnen zijn. De focus van het hoofdstuk ligt op intrinsieke vermenigvuldiging, i.e. het resultaat van het product is opnieuw een distributie. Omdat sommige producten onmogelijk zijn kunnen we enkel gedeeltelijke vermenigvuldigingsafbeeldingen definieren. We beginnen met de standaard Schwartz vermenigvuldiging. De Schwartz vermenigvuldiging definiëert een product tussen gladde functies en distributies via transpositie van de vermenigvuldigingsafbeelding $\mathcal{D} \to \mathcal{D} : \varphi \mapsto f\varphi$, voor $f \in C^{\infty}$. We zien dat zelfs het Schwartz product geen volledig continue vermenigvuldiging is. Door middel van localisatie kan de Schwartz vermenigvuldiging uitgebreid naar een product tussen distributies met disjuncte singuliere drager. Vervolgens bespreken we de dualiteitsmethode. Als een deelruimte van distributies X normaal is, dan bekomen we een product $X_{\text{loc}} \times X'_{\text{loc}} \to \mathcal{D}'$, waarbij $f \in \mathcal{D}'$ in X_{loc} is indien $f\varphi \in \mathcal{D}$ voor alle $\varphi \in \mathcal{D}$. Het Fourier product is gebaseerd op de $\mathcal{S}'$-convolutie. Het Fourier product bestaat indien de $\mathcal{S}'$-convolutie van de Fourier transformaties bestaat. Vervolgens bespreken we het strikt product en het model product. Deze producten zijn het meest algemeen en worden gedefiniëerd via regularisatie van de factoren. Regularisatie via strikte en model delta netten respectievelijk. Voor elk van deze producten geven we voorbeelden, bespreken we continuiteit. We eindigen het hoofdstuk met een korte bespreking van extrinsieke vermenigvuldiging. Het bekende onmogelijkheidsresultaat van Schwartz [33], zegt dat de ruimte van distributies niet uitgebreid kan worden tot een associatieve, commutatieve differentiealgebra waarbij de vermenigvuldiging van continue functies samenvalt met de puntsgewijze vermenigvuldiging. We bespreken kort de speciale Colombeau algebra $\mathcal{G}^s$. Dat is een algebra waarvoor vermenigvuldiging van gladde functies puntsgewijs is. Vervolgens bespreken we het concept van een zeer zwakke oplossing voor een partiele differentiaalvergelijking. De zeer zwakke oplossing werd voor het eerst gebruikt in het artikel [31]. Existentie en uniciteit van een zeer zwakke oplossing wordt uitgelegd aan de hand van een eenvoudig voorbeeld.

In het derde hoofdstuk definieren we een zeer zwakke oplossing van de Euler-Bernoulli vergelijking. De Euler-Bernoulli vergelijking is een partiële differentiaalvergelijking die de buiging van een balk onder verticale en axiale krachten beschrijft. De Euler-Bernoulli evenwichtsvergelijking wordt vaak gebruikt in de ingenieurswetenschappen om stabiliteit van een balk na te gaan. Wij bespreken de dynamische Euler-Bernoulli vergelijking die de beweging van de balk doorheen de tijd beschrijft. In het artikel 'Generalized solutions to the Euler-Bernoulli model with distributional forces' [21] beschrijven Oparnica en Hörmann een oplossing van de Euler-Bernoulli vergelijking in een Colombeau algebra. We gebruiken deze resultaten om een very weak solution van de Euler-Bernoulli vergelijking met distributionele krachten te beschrijven. Als eerste in het hoofdstuk beschrijven we hoe de Euler-Bernoulli vergelijking tot stand komt. Daarna definieren we een zeer zwakke oplossing van de Euler-Bernoulli vergelijking. We bewijzen existentie van de zeer zwakke oplossing. Onder voldoende regulariteit van de oplossing bewijzen we uniciteit. Tenslotte voeren we een numerieke analyse van de vergelijking uit. Met de eindige elementenmethode en regularisatie benaderen we de oplossing van de Euler-Bernoulli vergelijking met distributionele krachten. We bespreken enkele voorbeelden zoals een balk met discontinue doorsnede, een verticale puntkract en een balk met een breuk.

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